

## Appendix to "Debt Non-Neutrality, Policy Interactions, and Macroeconomic Stability"

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### 1 Properties of the steady state

The properties of the steady state are summarized in the following proposition.

**Proposition A1** *Assume that fiscal and monetary policy satisfy (15). Then a steady state of the model described in definition 1 exists and is uniquely determined. It is characterized by i)  $\bar{y} > 0$ ,  $\bar{\pi} \geq 1$ , and  $\bar{b} > 0$ ; ii)  $\partial\bar{y}/\partial\kappa = 0$ ,  $\partial\bar{\pi}/\partial\kappa < 0$ , and  $\partial\bar{b}/\partial\kappa \leq 0 \Leftrightarrow \rho_\pi \geq 1 - \Psi$ , where  $\Psi \equiv \frac{\bar{h}_{bb}}{1-\bar{h}_b} \frac{\bar{b}}{\bar{\pi}} > 0$ ; and iii)  $\partial\bar{y}/\partial\rho_\pi = 0$ ,  $\partial\bar{\pi}/\partial\rho_\pi > 0$ , and  $\partial\bar{b}/\partial\rho_\pi \geq 0 \Leftrightarrow \rho_\pi \geq 1 - (\Psi + \Upsilon)$ , where  $\Upsilon \equiv \frac{\bar{\pi}(1+\ln\bar{\pi})}{\partial\bar{\pi}/\partial\rho_\pi} > 0$ .*

**Proof.** The steady state condition  $\rho_\pi < \beta/(1 - \kappa)$  determines  $\bar{y}$  independently of the policy parameters, such that  $\partial\bar{y}/\partial\kappa = \partial\bar{y}/\partial\rho_\pi = 0$ . Condition (13) implies that  $\partial\bar{\pi}/\partial\kappa = (\bar{R} - 1)/G'(\bar{\pi}) < 0$  and  $\partial\bar{\pi}/\partial\rho_\pi = -[(1 - \kappa)\bar{R}\ln\bar{\pi}]/G'(\bar{\pi}) > 0$ , given that (15) ensures  $G'(\bar{\pi}) < 0$ . Condition (14) can then be used to derive the impact on  $\bar{b}$ . As  $\partial\bar{b}/\partial\kappa = (\partial\bar{b}/\partial\bar{\pi})(\partial\bar{\pi}/\partial\kappa)$  and  $(\partial\bar{b}/\partial\bar{\pi}) = \bar{b}/\bar{\pi} - \bar{\pi}(1 - \rho_\pi) / (\bar{h}_{bb}\beta\bar{R}) \geq 0 \Leftrightarrow \frac{\bar{h}_{bb}}{1-\bar{h}_b} \frac{\bar{b}}{\bar{\pi}} \geq (1 - \rho_\pi)$ , we can conclude that  $\partial\bar{b}/\partial\kappa \geq 0 \Leftrightarrow \rho_\pi \leq 1 - \Psi$ . From (14), we obtain  $\partial\bar{b}/\partial\rho_\pi = \bar{b}\bar{\pi}^{-1}(\partial\bar{\pi}/\partial\rho_\pi) + \bar{\pi}(\bar{h}_{bb}\beta\bar{R})^{-1}[\bar{\pi}(1 + \ln\bar{\pi}) - (1 - \rho_\pi)(\partial\bar{\pi}/\partial\rho_\pi)] \geq 0 \Leftrightarrow \rho_\pi \geq 1 - (\Psi + \Upsilon)$ .

■

Output and (equivalently) consumption are not affected by monetary or fiscal policy measures in the steady state. Assuming that (15) is satisfied, steady state inflation unambiguously rises with the reactivity of monetary policy and declines with a permanent rise in the fiscal policy parameter  $\kappa$  governing the proportion of tax financing. The effects on public debt are not unambiguous. A rise in  $\kappa$  leads to a decline in real public debt if and only if the inflation elasticity of the interest rate rule is sufficiently aggressive,  $\rho_\pi > 1 - \Psi$ . The latter has a further (direct) impact on public debt via (14), making real public debt increase with  $\rho_\pi$  for  $\rho_\pi > 1 - (\Psi + \Upsilon)$ . It should be noted that the results summarized in proposition A1 also apply for version B, where  $\rho_\pi < \frac{\beta(1-\delta)}{1-\kappa}$  replaces condition (15) and the composite parameter  $\Psi$  is defined as  $\Psi \equiv \frac{\delta'}{1-\delta} \frac{\bar{b}}{\bar{\pi}} > 0$ .

Summing up, while the fiscal authority can reduce nominal debt by raising the share of tax financing (see 10), its influence on the real value of outstanding debt crucially relies on the reaction of inflation and thus on the stance of monetary policy. Note that in the balanced budget case  $\kappa = 1$ , from (13) steady state inflation is  $\bar{\pi} = 1$ .

## 2 Saddle path stability

The deterministic version of the model (16)-(19) can be summarized as

$$\begin{pmatrix} \widehat{y}_{t+1} \\ \widehat{\pi}_{t+1} \\ \widehat{b}_t \end{pmatrix} = \begin{pmatrix} -(\sigma\beta)^{-1}\omega(\Psi - 1) + 1 & \Xi & -\Psi/\sigma \\ -\frac{1}{\beta}\omega & \frac{1}{\beta} & 0 \\ 0 & \eta\rho_\pi - 1 & 1 \end{pmatrix} \begin{pmatrix} \widehat{y}_t \\ \widehat{\pi}_t \\ \widehat{b}_{t-1} \end{pmatrix} = A \begin{pmatrix} \widehat{y}_t \\ \widehat{\pi}_t \\ \widehat{b}_{t-1} \end{pmatrix},$$

where  $\Xi \equiv \frac{1}{\sigma}(\rho_\pi - \Psi(\eta\rho_\pi - 1) + (\Psi - 1)/\beta)$ . Since there is one predetermined state variable ( $\widehat{b}_{t-1}$ ), while the other two variables can jump, a saddle path configuration obtains if the matrix  $A$  has exactly one eigenvalue with modulus smaller than one. The characteristic polynomial of  $A$  reads  $H(X) = X^3 - (\sigma\beta)^{-1}(\sigma + \omega + 2\sigma\beta - \Psi\omega)X^2 - (\sigma\beta)^{-1}(\Psi\eta\omega\rho_\pi - \omega - \sigma\beta - \omega\rho_\pi - 2\sigma)X - (\sigma\beta)^{-1}(\sigma + \omega\rho_\pi)$ . The determinant of  $A$  is strictly larger than one,  $\det(A) = -H(0) = (\sigma\beta)^{-1}(\sigma + \omega\rho_\pi) > 1$ , indicating that  $A$  exhibits at least one unstable eigenvalue. Given that

$$H(1) = (\sigma\beta)^{-1}(1 - \eta\rho_\pi)\Psi\omega,$$

there is at least one stable (and positive) eigenvalue lying between zero and one if  $1 > \eta\rho_\pi$ . As  $H(-1) = \omega(\sigma\beta)^{-1}[\Psi(1 + \eta\rho_\pi) - 2(1 + \rho_\pi) - 4\frac{\sigma}{\omega}(1 + \beta)]$ , we know that  $H(-1) < 0$  for  $\Psi \leq 2$  and  $\eta < 1$ , and that the third eigenvalue is unstable. Hence, the model exhibits exactly one stable and positive eigenvalue if and only if  $1 - \eta\rho_\pi > 0 \Leftrightarrow \rho_\pi < 1 + \frac{\kappa}{(1-\kappa)\bar{R}}$ . ■

It should be noted that the condition  $\rho_\pi < \beta/(1 - \kappa)$ , which ensures steady state uniqueness, is sufficient for saddle path stability: The existence of transaction services implies  $\bar{R} = \frac{\bar{\pi}}{(1-h_b)\beta}$  to hold in a steady state such that  $\bar{R} \leq \frac{\bar{\pi}}{\beta}$ , since  $h_b \leq 0$ . Further using that inflation satisfies  $\bar{\pi} = 1 + (1 - \kappa)(\bar{R} - 1)$  in the steady state, implying  $\beta \leq \frac{1+(1-\kappa)(\bar{R}-1)}{\bar{R}} \Leftrightarrow \beta/(1 - \kappa) \leq 1 + \frac{\kappa}{(1-\kappa)\bar{R}}$ , we can conclude that  $\rho_\pi < \beta/(1 - \kappa) \Rightarrow \rho_\pi < 1/\eta$ .

## 3 Solution of the sticky-price model

The fundamental solution of the sticky-price model features the state variables  $S_t = (\widehat{b}_{t-1}, \widehat{\varphi}_t)'$ . In what follows we assume that (15) is satisfied, such that the fundamental solution is the unique solution to (16)-(19). The model is solved applying the method of undetermined coefficients in  $\widehat{b}_t = \delta_b\widehat{b}_{t-1} + \delta_{bc}\widehat{\varphi}_t$ ,  $\widehat{\pi}_t = \delta_{\pi b}\widehat{b}_{t-1} + \delta_{\pi c}\widehat{\varphi}_t$ , and  $\widehat{x}_t = \delta_{xb}\widehat{b}_{t-1} + \delta_{xc}\widehat{\varphi}_t$ . Given that (15) is assumed to be satisfied, the equilibrium is saddle path stable such that  $\delta_b \in (0, 1)$ .

Hence, we aim at deriving the solutions for the remaining coefficients as functions of  $\delta_b$ . The two other coefficients describing the structural part of the solution are given by  $\delta_{\pi b} = (1 - \delta_b)/(1 - \eta\rho_\pi) > 0$ ,  $\delta_{xb} = (1 - \beta\delta_b)(1 - \delta_b)/[\omega(1 - \eta\rho_\pi)] > 0$ , which are unambiguously positive since  $\rho_\pi < \beta/(1 - \kappa)$  implies  $\rho_\pi < 1/\eta$  (see above). The coefficients on the cost-push shocks ( $\widehat{\varphi}_t$ ) are given by  $\delta_{bc} = -(1 - \eta\rho_\pi)(1 - \rho_c)\sigma/\Theta$ ,  $\delta_{\pi c} = (1 - \rho_c)\sigma/\Theta$ , and  $\delta_{xc} = -[\omega(\rho_\pi(1 - \eta\Psi) + 1 - (\delta_b + \rho_c)(1 - \Psi)) + \sigma(1 - \delta_b)(1 - \beta\delta_b)]/(\omega\Theta)$ , where  $\Theta \equiv -\omega[(\delta_b + \rho_c)(1 - \Psi) + \rho_\pi(\eta\Psi - 1) - 1] + \sigma[(1 - \beta\rho_c)(2 - \delta_b - \rho_c) + \beta(1 - \delta_b)^2]$ .

For the case where debt is neutral  $\Psi = 0$ , we get the standard New Keynesian model

$\sigma \hat{x}_t = \sigma E_t \hat{x}_{t+1} - \rho_\pi \hat{R}_t + E_t \hat{\pi}_{t+1}$ , and  $\hat{\pi}_t = \beta E_t \hat{\pi}_{t+1} + \omega \hat{x}_t + \hat{\varphi}_t$ . The fundamental solution exhibits no endogenous state variable and is known to be characterized by the following solution:  $\hat{x}_t = \tilde{\delta}_{xc} \hat{\varphi}_t$  and  $\hat{\pi}_t = \tilde{\delta}_{\pi c} \hat{\varphi}_t$ , where  $\tilde{\delta}_{xc} = -\frac{\rho_\pi - \rho_c}{\omega(\rho_\pi - \rho_c) + (1 - \beta \rho_c)(1 - \rho_c)\sigma}$  and  $\tilde{\delta}_{\pi c} = \frac{\sigma(1 - \rho_c)}{\omega(\rho_\pi - \rho_c) + (1 - \beta \rho_c)(1 - \rho_c)\sigma}$ .