A LeChatelier Principle for Relative Demand and Implications for Directed Technical Change

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This paper introduces a LeChatelier Principle for Relative Demand and applies it to the theory of directed technical change. The LeChatelier Principle for Relative Demand says that relative inverse demand curves are less elastic in the relative input quantity in the long run than in the short run. Applying this to general equilibrium models with endogenous technology, the paper proposes necessary and sufficient conditions for weak and strong relative equilibrium bias of technology. Weak relative equilibrium bias means that any increase in the relative supply of two input factors induces technical change biased towards the input that has become relatively more abundant. I prove that there is weak relative equilibrium bias if and only if a proportionate increase in the supply of the two inputs does not induce biased technical change. Strong relative equilibrium bias means that the relative input price is increasing in relative supply in the long run. I prove that there is strong relative equilibrium bias if and only if aggregate production is not jointly concave in the two inputs and technology when restricted to the isoquant. The results open the theory of directed technical change to new applications. The paper sketches potential applications to assignment models of the labor market, Ricardian models of international trade, and the theory of optimal taxation.

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1. Introduction

Models of directed technical change show that the endogenous response of technology to changes in the supply of input factors may substantially alter the relationship between input supply and input prices. In models with purely factor-augmenting technologies, an increase in the relative supply of input L_1 versus input L_2 induces an adjustment of technology (technical change) with a positive effect on the relative price of L_1 versus L_2 (biased towards L_1 versus L_2): there is *weak relative equilibrium bias* of technology. If and only if the elasticity of substitution between the two inputs exceeds a certain threshold, the effect of the induced technical change overcompensates the negative direct effect of the increase in relative supply, such that the relative factor price of L_1 versus L_2 increases in relative supply in the long run: there is *strong relative equilibrium bias* of technology if and only if the elasticity of substitution is sufficiently large. These results were first presented by Acemoglu (1998) and Kiley (1999) and have been applied to several practical questions in economics thereafter.[1]

In many situations, there is, however, no justification, neither empirical nor theoretical, for assuming technology to be purely factor-augmenting. In several potential applications it may even be impossible to make this assumption, either because the analyzed technology has a specific but not factor-augmenting form, or because the allocation of production inputs to positions in the production process is endogenous and an explicit solution for aggregate production does not exist.[2] Without factor-augmenting technologies, conditions for relative equilibrium bias are unknown.[3]

It is therefore important for at least three reasons to have a general theory of relative equilibrium bias: first, to learn whether the assumption of factor-augmenting technology is critical in existing applications; second, to enlarge the scope of the theory of directed technical change to situations incompatible with pure factor-augmentation; third, to learn about the limits of the theory.

In this paper, I propose such a general theory of relative equilibrium bias and discuss

[1] Applications include: work on the sources of growing wage inequality in industrialized countries (Acemoglu 1998 and Kiley 1999), on the effect of international trade on wage inequality and growth (Acemoglu 2003b and Bonafiglioli and Gancia 2008), and on optimal policies to reduce carbon dioxide emissions (Acemoglu et al. 2012). See also the paragraph Related Literature at the end of this section.

[2] I give examples for such models in Section 3.3.

[3] There exists however a general theory of absolute equilibrium bias of technology, proposed by Acemoglu (2007). Analogous to the definitions of relative equilibrium bias, weak absolute equilibrium bias means that an increase in the supply of an input induces technical change with a positive effect on the price of the input, while strong absolute equilibrium bias means that the effect of the induced technical change overcompensates the negative direct effect of the increase in supply, such that the input price increases in input supply in the long run.

several applications, which could not be treated convincingly on the basis of existing results.

To set up the general theory, I first derive a result about the shape of relative inverse demand curves, stating that relative inverse demand is less elastic in the long run than in the short run. Due to its close analogy to the LeChatelier Principle (see Samuelson 1947 and Milgrom and Roberts 1996), which gives the same result for absolute inverse demand curves, I call the new result the LeChatelier Principle for Relative Demand. The new principle is a relevant extension of the theory of demand in itself.

Applying the LeChatelier Principle for Relative Demand first to a class of static, then to a class of dynamic general equilibrium models with endogenous technology yields general necessary and sufficient conditions for weak relative equilibrium bias. These conditions are much weaker than the condition that technology is purely factor-augmenting. I complement the conditions by deriving necessary and sufficient conditions for strong relative equilibrium bias, which are similar to those obtained by Acemoglu (2007) for strong absolute equilibrium bias (see footnote 3).

Finally, I discuss how to apply the developed theory to support existing applications of directed technical change and to analyze endogenous technology in assignment models of the labor market, Ricardian models of international trade, models of optimal income taxation, and models of automation or factor-replacing technologies.

In more detail, the LeChatelier Principle for Relative Demand studies properties of the solutions to the problem $\max_{(\theta,L)} \{F(\theta, L) - p^T L\}$, when θ is fixed in the short run and variable in the long run. It says that any increase in the relative quantity of two inputs, say L_1/L_2, induces an adjustment of long-run inputs θ with a positive effect on relative inverse demand for L_1 versus L_2 if and only if a proportionate increase in L_1 and L_2 does not change relative inverse demand via the induced adjustment of long-run inputs.[4,5] In addition, an increase in L_1/L_2 that leaves (L_1, L_2) on the current isoquant always induces an adjustment of long-run inputs with a positive effect on relative inverse demand. If relative inverse demand is homogeneous of degree zero in L_1 and L_2 – such that it can be written as a function of the relative quantity L_1/L_2 – it is less elastic in the relative quantity in the long run than in the short run.

I first provide a local version of the principle, which applies to infinitesimal changes

[4] Another way to formulate the condition is that the difference between long-run and short-run relative inverse demand must be homogeneous of degree zero in L_1 and L_2.

[5] Note that we are interested in relative inverse demand here, so we do comparative statics with respect to the relative quantity L_1/L_2, which is a solution to the maximization problem itself. The question we ask is hence: how must the relative price p_1/p_2 change between the short run and the long run if the relative quantity L_1/L_2 that is to solve the maximization problem (both in the short and the long run) changes.

in input quantities when the objective function is differentiable. Then, using the theory of monotone comparative statics, I give a generalization of the local version, which covers discrete changes in input quantities and neither requires the objective to be differentiable (in θ) nor long-run inputs to live on a smooth manifold. Importantly, the generalized version applies to problems with discrete choice spaces, which include many natural settings such as a firm's problem of choosing between different organizational forms or between different technologies without a natural measure of distance between them.

The intuition behind the LeChatelier Principle for Relative Demand is best understood via its relation to the original LeChatelier Principle, which says that absolute inverse demand is less elastic in the long run than in the short run.[6] A simple example is given by the neoclassical model of a firm demanding labor L and capital θ. When L increases, inverse demand for labor, equal to its marginal product, falls. Since labor is q-complementary to capital, the return to capital increases and the firm raises its capital stock in the long run. This, again via q-complementarity between capital and labor, has a positive effect on inverse demand for labor, such that inverse demand for labor falls by less in the long run than in the short run. The result is different for relative inverse demand. Considering a firm with two types of labor, L_1 and L_2, and capital, an increase in L_1 still induces the firm to raise the capital stock in the long run. There is, however, no argument preventing that capital is more complementary to L_2 than to L_1 (meaning that it increases the marginal product of L_2 by more than that of L_1 in relative terms). The increase in capital may therefore have a negative, rather than a positive, effect on relative inverse demand for L_1 versus L_2, which is equal to the ratio of their marginal products.

Yet, when restricting attention to input changes along the isoquant, a clear result emerges. Let the increase in L_1 be accompanied by a decrease in L_2 that exactly offsets the output expansion from the increase in L_1. Such a change increases the marginal product of capital if and only if capital is more complementary to L_1 than to L_2.[7] The adjustment of capital will therefore always have a positive effect on relative inverse demand for L_1 versus L_2. With the observation that any increase in L_1/L_2

[6]The more common formulation of the original LeChatelier Principle applies to direct, as opposed to inverse, demand curves and says that direct demand is more elastic in the long run than in the short run. The two formulations are equivalent.

[7]To see this, note that the marginal effect of the increase in L_1 on the marginal product of θ is given by $D_{\theta L_1}F(\theta, L)$. The effect of the decrease in L_2 is given by $d_I D_{\theta L_2}F(\theta, L)$, where d_I is the marginal change in L_2 required to offset the output effect of the increase in L_1: $d_I = -D_{L_1}F(\theta, L)/D_{L_2}F(\theta, L)$. The total effect on the marginal product of θ is positive if and only if $D_{\theta L_1}F(\theta, L) + d_I D_{\theta L_2}F(\theta, L) \geq 0$. But this holds if and only if the effect of an increase in θ on relative inverse demand for L_1 versus L_2

can be decomposed into a change along the isoquant and a proportionate change in L_1 and L_2, the result holds for any increase in L_1/L_2 if and only if a proportionate increase does not induce an adjustment of capital – at least not one that has an effect on relative inverse demand. The two versions of the LeChatelier Principle for Relative Demand in Section 2 make these arguments formal in settings where (i) long-run inputs θ are allowed to be multidimensional, and (ii) the objective F is allowed to be non-differentiable in θ.

Applying the LeChatelier Principle for Relative Demand to static general equilibrium models shows that weak relative equilibrium bias is a prevalent phenomenon in a wide class of endogenous technology models.[8] In particular, within this class of models, there is weak relative equilibrium bias of technology if and only if the difference between long-run and short-run relative input prices is homogeneous of degree zero in the two inputs. Moreover, there is always weak relative equilibrium bias when restricting supply changes to the isoquant. In addition, I prove that there is strong relative equilibrium bias if and only if the Hessian of the objective function is not negative semi definite on the isoquant. This precludes strong relative equilibrium bias in models where the firms that demand non-technology inputs choose their technologies themselves. Neither of the results makes any reference to factor-augmenting technologies.

The conditions for static models transfer directly to a class of dynamic models, in which technology on the balanced growth path can be represented as an appropriately constrained maximizer of aggregate output. I show that this condition is satisfied in a generalized version of the standard dynamic model of directed technical change.[9] The generalized model only imposes restrictions on aggregate production that are essential for existence of a unique balanced growth path. In particular, technology may not be factor-augmenting.

The developed theory has three important implications for applied work. First, weak relative equilibrium bias arises under weak conditions in a wide class of models. The assumption of purely factor-augmenting technologies can therefore be relaxed in presumably many existing applications. As an example, consider the explanation of the

is positive (assuming F is C^2):

$$D_\theta \left(D_{L_1}F(\theta,L)/D_{L_2}F(\theta,L)\right) = \left(D_{\theta L_1}F(\theta,L) + d_I D_{\theta L_2}F(\theta,L)\right)/D_{L_2}F(\theta,L) \geq 0.$$

[8]The class of models includes, among others, all models used by Acemoglu (2007) to analyze absolute equilibrium bias of technology.

[9]By the standard dynamic model of directed technical change I refer to the model proposed by Acemoglu (1998) and Kiley (1999).

joint upwards trend in the skill premium and the relative supply of skills via directed technical change, as proposed by Acemoglu (1998) and Kiley (1999). The dynamic model analyzed in Section 3.2 shows that the assumption of factor-augmenting technologies can be relaxed substantially in their models, lending support to the proposed explanation.

Second, the broad scope of the derived results makes new potential applications accessible to the theory of directed technical change – especially those incompatible with pure factor-augmentation. I sketch such applications in assignment models of the labor market, Ricardian models of international trade, and the theory of optimal taxation in Section 3.3. In all of these models it is necessary to move beyond the assumption of factor-augmenting technologies. In assignment models the allocation of workers to tasks is endogenous and explicit solutions for aggregate production do not exist in general. In Ricardian models of international trade I demonstrate how to endogenize transport technologies, which determine the cost of trade and have no factor-augmenting representation. In the theory of optimal taxation the restriction to factor-augmenting technologies would reduce the credibility of the normative results.

Third, weak relative equilibrium bias has precise limits: if and only if a proportionate increase in the supply of the two inputs under consideration induces biased technical change, weak relative equilibrium bias fails. I show that this is likely the case in models with technologies that directly replace some production factor, such as models of task automation.

The paper is organized as follows: the next paragraph gives an overview over related research. Section 2 presents the LeChatelier Principle for Relative Demand. Section 3.1 applies the principle to static general equilibrium models, and Section 3.2 transfers the results to dynamic models of directed technical change. I discuss the relevance of the developed theory for applied work and sketch potential future applications in Section 3.3. The appendix contains details on the classes of static and dynamic general equilibrium models used in Section 3 and the results from monotone comparative statics applied in Section 2.2.

Related Literature The paper is related to two branches of research in economics: research on directed technical change and research on the LeChatelier Principle.

Research on directed technical change comes in two waves. The first wave consists of the work on induced innovation from the 1960s, most prominently the seminal work by Kennedy (1964) and important contributions by Drandakis and Phelps (1966) and Nordhaus (1973). The second wave starts with the contributions of Acemoglu

(1998) and Kiley (1999). The models of the second wave are extensions of the R&D based endogenous growth models developed by Romer (1990), Grossman and Helpman (1991), Aghion and Howitt (1992), and Jones (1995) in the 1990s. This paper is concerned with models of the second wave. Whenever referring to models of directed technical change, I refer to models of the second wave. Whenever referring to the standard (dynamic) model of directed technical change, I refer to the seminal models by Acemoglu (1998) and Kiley (1999).[10] These models consist of a production sector with CES production function in two inputs, factor-augmenting technologies that are improved by the output of a research sector, and a standard representative household. The total amount of research and its distribution across technologies (its direction) are endogenous and respond to profit incentives. Profit incentives, in turn, depend on the marginal contributions of potential innovations to aggregate final goods production.

Acemoglu (1998) and Kiley (1999) use their models to propose an explanation for the joint increase of the relative supply of skilled versus unskilled workers and the corresponding relative wage in the United States since 1980. Thereafter, a series of papers applies the standard model to: (i) the effect of international trade (Acemoglu 2003b, Thoenig and Verdier 2003) and offshoring (Acemoglu et al. 2015) on wage inequality; (ii) productivity differences between industrialized and developing countries (Acemoglu and Zilibotti 2001, Bonafiglioli and Gancia 2008); (iii) climate change and the transition from fossil to renewable energy sources (Acemoglu et al. 2012); (iv) balanced growth in the neoclassical growth model (Acemoglu 2003a).

Analyses of directed technical change beyond settings with purely factor-augmenting technologies are contained in Acemoglu (2007) and Acemoglu (2010). Both articles are concerned with absolute, rather than relative, prices. Acemoglu (2007) notes the close relation between part of his results and the LeChatelier Principle. Indeed, the results in Section 3.1 of the present paper are the analogs for relative factor prices to the results for absolute factor prices in Acemoglu (2007) (as the LeChatelier Principle for Relative Demand presented in Section 2 is the analog for relative demand to the original LeChatelier Principle for absolute demand).

The first formal treatment of the LeChatelier Principle in economics is by Samuelson (1947). Alternative derivations and extensions are given by Samuelson (1960), Silberberg (1971), and Silberberg (1974). More recently, Milgrom and Roberts (1996), and Suen et al. (2000) provide generalizations of the LeChatelier Principle. Milgrom and

[10] The microfoundation in Acemoglu (1998) follows the creative destruction approach of Aghion and Howitt (1992), whereas the microfoundation in Kiley (1999) follows the increasing varieties approach of Romer (1990). Both microfoundations yield equivalent results in the present context. I refer to the creative destruction microfoundation as the standard microfoundation.

Roberts (1996) use results from Milgrom and Shannon (1994) on monotone comparative statics, rooted in lattice theory.

All mentioned articles are concerned with absolute, rather than relative, demand curves. I obtain the first version of the LeChatelier Principle for Relative demand in the setting studied by Samuelson (1947) (a smooth optimization problem with regular solutions). For the generalized version, I follow the generalization of Milgrom and Roberts (1996), using results from lattice theory in Milgrom and Shannon (1994).

2. The LeChatelier Principle for Relative Demand

In this section, I define a generic (profit or utility) maximization problem, derive relative inverse demand functions, and prove two versions of the LeChatelier Principle for Relative Demand. The first version is a local result relying on differential calculus, the second version generalizes the first version to a global result using the theory of monotone comparative statics.

The environment is given by an agent maximizing the objective function $F(\theta, L) - p^T L$ with respect to (θ, L) subject to $\theta \in \Theta$, a compact partially ordered set, $L \in \mathcal{L} \subset \mathbb{R}^M_{++}$, and with p a strictly positive M-dimensional vector of factor prices. F is assumed to be continuous and differentiable in L with strictly positive first-order partial derivatives. I summarize:

Assumption 1. *The set Θ is partially ordered and compact in the interval topology;*[11] *$\mathcal{L}$ is a subset of $\mathbb{R}^M_{++}$, $M \geq 2$. The function $F : \Theta \times \mathbb{R}^M_{++} \to \mathbb{R}_+$, $(\theta, L) \mapsto F(\theta, L)$, is continuous in (θ, L) and differentiable in L. The first order partial derivatives of F with respect to L are strictly positive: $D_{L_i} F(\theta, L) > 0$ for $i = 1, 2, ..., M$ and any $(\theta, L) \in \Theta \times \mathbb{R}^M_{++}$.*

The local and global results require different additional assumptions on the structure of Θ. Such assumptions will thus be imposed later, for each of the results separately.

The agent can choose only L in the short run and both L and θ in the long run. For brevity, I refer to L as short-run inputs and to θ as long-run inputs. Short-run demand is defined as

$$L^{\star\star}(\theta, p) \equiv \underset{L \in \mathcal{L}}{\operatorname{argmax}} \left\{ F(\theta, L) - p^T L \right\}$$

[11] The interval topology results from taking all sets of the form $\{\theta \in \Theta | \theta \leq \bar{\theta}\}$ or $\{\theta \in \Theta | \theta \geq \bar{\theta}\}$ with $\bar{\theta} \in \Theta$ as a subbasis for the closed sets (Frink, 1942). For all local results, Θ is a compact subset of $\mathbb{R}^N$. In that case, the interval topology is equivalent to the Euclidean topology on Θ (Birkhoff, 1967).

for a given θ. Long-run demand is defined as

$$(L^\star, \theta^\star)(p) \equiv \underset{(L,\theta)\in\Theta\times\mathcal{L}}{\operatorname{argmax}} \left\{ F(\theta, L) - p^T L \right\}.$$

Whenever a maximizer $L^{\star\star}(\theta, p)$ is interior, first order conditions must hold:

$$D_{L_i} F(\theta, L^{\star\star}(\theta, p)) = p_i$$

for all i. This leads to inverse demand for L_i: short-run inverse demand for L_i as a function of L and θ is defined as $D_{L_i} F(\theta, L)$; long-run inverse demand for L_i as a function of L is defined as $D_{L_i} F(\theta^\star(L), L)$ with $\theta^\star(L) = \sup \operatorname{argmax}_{\theta\in\Theta} F(\theta, L)$. Note that $\operatorname{argmax}_{\theta\in\Theta} F(\theta, L)$ is non-empty by Assumption 1. In all local theorems, I assume that $\theta^\star(L)$ is the unique maximizer of $F(\theta, L)$ such that it necessarily exists. In all global theorems, the conditions imposed guarantee that $\operatorname{argmax}_{\theta\in\Theta} F(\theta, L)$ is a complete lattice such that the supremum $\theta^\star(L)$ is guaranteed to exist (in $\operatorname{argmax}_{\theta\in\Theta} F(\theta, L)$) as well.

Denote the images of short-run and long-run demand by

$$\mathcal{L}^s = \left\{ L \in \mathbb{R}^M_{++} | \exists \theta \in \Theta, p \in \mathbb{R}^M_{++} : L \in L^{\star\star}(\theta, p) \right\}$$

and

$$\mathcal{L}^l = \left\{ L \in \mathbb{R}^M_{++} | \exists \theta \in \Theta, p \in \mathbb{R}^M_{++} : (L, \theta) \in (L^\star, \theta^\star)(p) \right\}.$$

On $\mathcal{L}^s \times \Theta$ and $\mathcal{L}^l$, short-run and long-run inverse demand are the inverse functions of the two demand correspondences. Yet, for most of the following analysis the restriction of inverse demand to $\mathcal{L}^s \times \Theta$ and $\mathcal{L}^l$ is irrelevant, so, for expositional simplicity, I impose it only where necessary.

The paper is concerned with relative inverse demand. I define:

Definition 1. *Short-run relative inverse demand* for L_1 versus L_2 is defined as

$$p^s_{12}(\theta, L) \equiv \frac{D_{L_1} F(\theta, L)}{D_{L_2} F(\theta, L)}.$$

Long-run relative inverse demand for L_1 versus L_2 is defined as

$$p^l_{12}(L) \equiv \frac{D_{L_1} F(\theta^\star(L), L)}{D_{L_2} F(\theta^\star(L), L)},$$

where $\theta^\star(L) \equiv \sup \operatorname{argmax}_{\theta\in\Theta} F(\theta, L)$.

Note that $p^l_{12}(L) \equiv p^s_{12}(\theta^\star(L), L)$ and that the selection of L_1 and L_2 from the input vector L is arbitrary.

The isoquants of F with respect to (L_1, L_2) play a central role in the derivation of the LeChatelier Principle for Relative Demand. For clarity, I define:

Definition 2. The *isoquant of F at* $(\bar{\theta}, \bar{L})$ *with respect to* (L_1, L_2) is defined as $I_{12}(\bar{\theta}, \bar{L}) \equiv \{(L_1, L_2) \in \mathbb{R}^2_{++} | F(\bar{\theta}, L_1, L_2, \bar{L}_{-12}) = F(\bar{\theta}, \bar{L})\}$, where $\bar{L}_{-12} = (\bar{L}_3, \bar{L}_4, ..., \bar{L}_M)^T$. The function $L_2(L_1; \bar{\theta}, \bar{L})$ is defined such that $\big(L_1, L_2(L_1; \bar{\theta}, \bar{L})\big) \in I_{12}(\bar{\theta}, \bar{L})$.

Note that the isoquant is defined at fixed long-run input levels $\bar{\theta}$, so $\bar{\theta}$ enters the function $L_2(L_1; \bar{\theta}, \bar{L})$ as a parameter only.[12]

2.1. The Local Principle

The local version of the LeChatelier Principle for Relative Demand states that long-run relative inverse demand is less elastic than short-run relative inverse demand. To make use of differential calculus, I assume:

Assumption 2. *The set* Θ *is a subset of* $\mathbb{R}^N$. *The function* $F : \Theta \times \mathbb{R}^M_{++} \to \mathbb{R}_+$, $(\theta, L) \mapsto F(\theta, L)$, *is* C^2 *on the interior of its domain.*

With all definitions and assumptions in place, I can state the local version of the LeChatelier Principle for Relative Demand. The proof of the theorem is provided and discussed in the paragraph Proof of the Local Principle at the end of the present subsection.

Theorem 1. *Let F and* Θ *satisfy Assumptions 1 and 2. Let* $\bar{L} \in \mathbb{R}^M_{++}$ *and suppose that* $\theta^\star(\bar{L}) \in \Theta^{\circ}$[13] *is a regular and unique maximizer of* $F(\theta, \bar{L})$ *on* Θ. *Denote* $\theta^\star(\bar{L})$ *by* $\bar{\theta}$, $D_{L_1} L_2(L_1; \bar{\theta}, \bar{L})$ *by* d_I, *the directional derivative* $v_1 D_{L_1} p^{s(l)}_{12}(\cdot) + v_2 D_{L_2} p^{s(l)}_{12}(\cdot)$ *by* $D_{(L_1,L_2)=(v_1,v_2)} p^{s(l)}_{12}(\cdot)$, *and the set of direction vectors which increase* L_1/L_2 *at* $\bar{L}$ *by* $\bar{V} = \{(v_1, v_2) \in \mathbb{R}^2 | v_1/\bar{L}_1 > v_2/\bar{L}_2\}$. *Then:*

1.
$$D_{(L_1,L_2)=v} p^s_{12}(\bar{\theta}, \bar{L}) \leq D_{(L_1,L_2)=v} p^l_{12}(\bar{L}) \quad \forall v \in \bar{V}$$

(with strict inequality if and only if $D_{(L_1,L_2)=(1,d_I)} \theta^\star(\bar{L}) \neq \mathbf{0}$*)*
if and only if $p^l_{12} - p^s_{12}$ *is locally homogeneous of degree zero in* (L_1, L_2) *at* $\bar{L}$, *that is* $D_{(L_1,L_2)=(\bar{L}_1,\bar{L}_2)} \left[p^l_{12}(\bar{L}) - p^s_{12}(\bar{\theta}, \bar{L})\right] = 0.$

[12] The definition of an isoquant corresponds to the notion of a short-run isoquant: when long-run inputs are fixed, changes in L_1 and L_2 along the isoquant leave production unchanged. This notion is different from that of a long-run isoquant: on a long-run isoquant, production is unchanged when the long-run inputs adjust according to $\theta = \theta^\star(L)$ to changes in L_1 and L_2.

[13] I use the notation Θ° to denote the interior of the set Θ.

2.
$$D_{(L_1,L_2)=(1,d_l)}p^s_{12}(\bar\theta,\bar L) \le D_{(L_1,L_2)=(1,d_l)}p^l_{12}(\bar L)$$
with strict inequality if and only if $D_{(L_1,L_2)=(1,d_l)}\theta^\star(\bar L) \neq \mathbf{0}$.

The theorem makes statements about how the adjustment of long-run inputs to changes in the quantities L_1 and L_2 affects relative inverse demand. This effect is conveniently summarized in the effect of changes in (L_1, L_2) on the difference between long-run and short-run relative inverse demand, $p^l_{12} - p^s_{12}$: if a change in (L_1, L_2) increases (decreases) $p^l_{12} - p^s_{12}$, it induces an adjustment of long-run inputs with a positive (negative) effect on relative inverse demand.

The main result of the theorem is that, whenever a proportionate change in (L_1, L_2) has no effect on $p^l_{12} - p^s_{12}$ (induces an adjustment of long-run inputs with no effect on relative inverse demand), any increase in the ratio L_1/L_2 will increase $p^l_{12} - p^s_{12}$ (will induce an adjustment of long-run inputs with a positive effect on relative inverse demand).

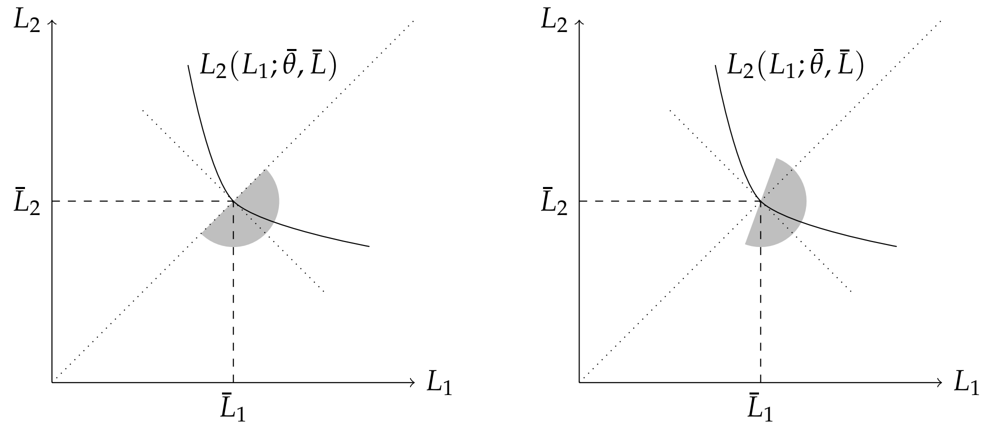
The main result also has a compact interpretation in terms of demand elasticities: whenever relative inverse demand is homogeneous of degree zero in (L_1, L_2) in the short and the long run, or, put differently, whenever relative inverse demand is a well-defined function of the ratio L_1/L_2 (instead of the levels L_1 and L_2), then relative inverse demand is less elastic (in L_1/L_2) in the long run than in the short run. I emphasize this interpretation as a separate corollary:

Corollary 1. *Take $\bar L \in \mathcal{L}^l$ (such that $(\bar L, \bar\theta)$ is in the image of the demand function $(L^\star, \theta^\star)$) and suppose that the conditions of Theorem 1 are satisfied. In addition, suppose that relative inverse demand is homogeneous of degree zero in (L_1, L_2) in the short and long run. Write relative inverse demand as a function of l_{12} and denote the elasticity of relative inverse demand with respect to l_{12} by $\epsilon^{s(l)}_{p,l}(\cdot) \equiv |D_{l_{12}}p^{s(l)}_{12}(\cdot)l_{12}/p^{s(l)}_{12}(\cdot)|$. Then, relative inverse demand is less elastic in the long run than in the short run at $\bar L$:*
$$\epsilon^l_{p,l}(\bar\theta,\bar L) \le \epsilon^s_{p,l}(\bar L).$$

Proof. Since $(\bar L, \bar\theta)$ is in the image of the demand function $(L^\star, \theta^\star)$, relative inverse demand is decreasing in l_{12} at $\bar L$ in the long run, $D_{l_{12}}p^l_{12}(\bar l_{12}, \bar L_{-12}) \le 0$.[14] Then, the corollary follows immediately from Theorem 1. □

Besides its main result, Theorem 1 is also informative about the case when $p^l_{12} - p^s_{12}$

[14]For a formal proof of this claim, note that the Hessian of F must be negative semi definite at $(\bar L, \bar\theta)$ and apply the proof of Theorem 3 in Section 3.1.



(a) The isoquant $I_{12}(\bar{\theta}, \bar{L})$ and the range of direction vectors, starting at $(\bar{L}_1, \bar{L}_2)$, which increase $p^l_{12} - p^s_{12}$ (colored gray) for the case that $p^l_{12} - p^s_{12}$ is locally homogeneous of degree zero in (L_1, L_2).

(b) The isoquant $I_{12}(\bar{\theta}, \bar{L})$ and the range of direction vectors, starting at $(\bar{L}_1, \bar{L}_2)$, which increase $p^l_{12} - p^s_{12}$ (colored gray) for the case that a (marginal) proportional increase in (L_1, L_2) increases $p^l_{12} - p^s_{12}$.

Figure 1

is not homogeneous of degree zero in (L_1, L_2). First, part 1 is a biconditional statement. So, whenever $p^l_{12} - p^s_{12}$ is not (locally) homogeneous of degree zero in (L_1, L_2), there exists a direction to change (L_1, L_2) which increases L_1/L_2 but decreases $p^l_{12} - p^s_{12}$. Put differently, there exists a direction along which relative inverse demand is more elastic in the long run than in the short run. Second, part 2 says that an increase in L_1/L_2 in direction of the corresponding isoquant always increases $p^l_{12} - p^s_{12}$. Put differently, relative inverse demand is always less elastic in the long run than in the short run along the isoquant $I_{12}(\bar{\theta}, \bar{L})$. Note that this statement does not require any assumptions in addition to the differentiability requirements in Assumption 2.

Figure 1 illustrates the results. In Figure 1a, $p^l_{12} - p^s_{12}$ is (locally) homogeneous of degree zero in (L_1, L_2). Hence, changes of (L_1, L_2) in any direction that increases L_1/L_2 will increase $p^l_{12} - p^s_{12}$. All such directions are colored gray. In Figure 1b, a proportionate increase in (L_1, L_2) has a positive effect on $p^l_{12} - p^s_{12}$. Hence, there exist directions which increase L_1/L_2 but decrease $p^l_{12} - p^s_{12}$. This holds in particular for directions close to a proportionate decrease in (L_1, L_2). Again, all directions which increase $p^l_{12} - p^s_{12}$ are colored gray. Note that part 2 of the theorem says that the gray range must always include the direction of the isoquant (towards L_1), indicated by the dotted tangent line.

The intuition for the result is as follows: increasing L_1 and decreasing L_2 to hold output constant increases the marginal product of long-run inputs which are (i) q-complementary to L_1 but not to L_2, or (ii) q-complementary to L_1 and L_2, but more so to L_1 (in the sense that they have a positive effect on the marginal product of L_1 relative to L_2).[15] Hence, changes in L_1 and L_2 along the isoquant induce the optimizing agent to increase those long-run inputs that have a positive effect on relative inverse demand for L_1 versus L_2. Proportional changes in L_1 and L_2, on the other hand, increase the marginal product of long-run inputs which are (i) q-complementary to both L_1 and L_2, or (ii) sufficiently q-complementary to one of the two inputs and not q-complementary to the other. Such long-run inputs may have a positive or a negative effect on relative inverse demand.

The effect of a change of (L_1, L_2) in arbitrary direction, finally, is the weighted average of the effects of a change along the isoquant and a proportionate change: the closer the direction comes to the isoquant, the more the effect on long-run inputs resembles that of a change along the isoquant (which is to increase long-run inputs with a positive effect on relative inverse demand), and the closer the direction comes to a proportionate change, the more the effect resembles that of a proportionate change (which may be to increase long-run inputs with a negative effect on relative inverse demand).

Application of the Local Principle To apply (the arguably most important) part 1 of the theorem, one must know whether relative inverse demand is homogeneous of degree zero in (L_1, L_2) or not. Since this may not always be easy to check, it is desirable to have conditions on the objective F which ensure zero homogeneity of relative inverse demand. I derive such conditions for both the requirements that $p^l_{12} - p^s_{12}$ is homogeneous of degree zero in (L_1, L_2) and that p^l_{12} and p^s_{12} are homogeneous of degree zero in (L_1, L_2) individually. The former is used in Theorem 1, the latter in Corollary 1.

For zero homogeneity of the difference $p^l_{12} - p^s_{12}$ in (L_1, L_2) it is sufficient that the maximizer $\theta^\star$ is homogeneous of degree zero in (L_1, L_2). We can immediately ensure this by requiring F to be linear homogeneous in (L_1, L_2). A substantially weaker but still sufficient requirement is that θ enters F only via an inner function f with f linear homogeneous in (L_1, L_2) and the outer function strictly increasing in f.[16]

[15] If a given long-run input $\theta \in \mathbb{R}$ is q-complementary both to L_1 and L_2, the total effect on its marginal product is $D^2_{\theta L_1} F(\bar{\theta}, \bar{L}) + d_I D^2_{\theta L_2} F(\bar{\theta}, \bar{L})$. But this is equal to $D_{L_2} F(\bar{\theta}, \bar{L}) \left[D_\theta p^s_{12}(\bar{\theta}, \bar{L}) \right]$ and hence equal in sign to the effect of θ on relative inverse demand. So, the change in L_1 and L_2 induces an increase in θ if and only if an increase in θ has a positive effect on relative inverse demand.

[16] We can equivalently require the inner function f to be homogeneous of degree zero in (L_1, L_2).

Maximizing F with respect to θ is then equivalent to maximizing the inner function f with respect to θ, and since f is linear homogeneous in (L_1, L_2), the maximizer $\theta^\star$ will be homogeneous of degree zero in (L_1, L_2).

Condition 1. The function F can be written as a composition of functions g and f, $F(\theta, L) \equiv g(f(\theta, L), L)$, with $f : \Theta \times \mathbb{R}^M_{++} \to \mathbb{R}_+$ linear homogeneous in (L_1, L_2) and $g : \mathbb{R}_+ \times \mathbb{R}^M_{++} \to \mathbb{R}_+$ strictly increasing in its first argument.

To ensure in addition that p^l_{12} and p^s_{12} are homogeneous of degree zero in (L_1, L_2) individually, also the partial effect of proportionate changes in (L_1, L_2) on relative inverse demand (which is common to p^l_{12} and p^s_{12} and hence does not affect their difference) must be zero. For that, I augment Condition 1 by requiring the outer function g to be independent of (L_1, L_2). Thereby, relative inverse demand becomes identical to the ratio of partial derivatives of the inner function f; this ratio is homogeneous of degree zero in (L_1, L_2), because f is linear homogeneous in (L_1, L_2).

Condition 2. The function F can be written as a composition of functions g and f, $F(\theta, L) \equiv g(f(\theta, L), L_{-12})$, with $f : \Theta \times \mathbb{R}^M_{++} \to \mathbb{R}_+$ linear homogeneous in (L_1, L_2) and $g : \mathbb{R}_+ \times \mathbb{R}^{M-2}_{++} \to \mathbb{R}_+$ strictly increasing in its first argument $f(\theta, L)$.

I summarize:

Lemma 1. *Let F and Θ satisfy Assumption 1. Then:*

1. *If F satisfies Condition 1, then $\theta^\star$ (and thereby $p^l_{12} - p^s_{12}$) is homogeneous of degree zero in (L_1, L_2).*
2. *If F satisfies Condition 2, then p^s_{12} and p^l_{12} are homogeneous of degree zero in (L_1, L_2).*

Remark 1. Condition 2 is more restrictive than homotheticity, but less restrictive than homogeneity of F in (L_1, L_2).

Homotheticity allows the outer function g to be dependent on θ. Therefore, it does not guarantee that $\theta^\star$ is homogeneous of degree zero in (L_1, L_2). It follows that p^l_{12} is not necessarily homogeneous of degree zero in (L_1, L_2) under homotheticity.

Homogeneity of degree $\delta > 0$ requires g to be of the form

$$g(f(\theta, L), L_{-12}) = [f(\theta, L)]^\delta \, \tilde{g}(L_{-12}).$$

Therefore, it implies Condition 2 and the results of Lemma 1 apply under homogeneity.

Remark 2. Conditions 1 and 2 are unnecessarily restrictive in requiring θ to enter F only via the inner function f: it is sufficient that only those components of θ enter F exclusively via the inner function that have a non-zero effect on relative inverse demand.

Formally, suppose that F can be written as $F(\theta, L) \equiv g(f(\theta^{(1)}, L), \theta^{(2)}, L)$ with $\theta^{(1)} = (\theta_1, \theta_2, ..., \theta_k)^T$ and $\theta^{(2)} = (\theta_{k+1}, \theta_{k+2}, ..., \theta_N)^T$, such that $f : \Theta^{(1)} \times \mathbb{R}^M_{++} \to \mathbb{R}_+$ is linear homogeneous in (L_1, L_2), $g : \Theta^{(2)} \times \mathbb{R}^M_{++} \to \mathbb{R}_+$ is strictly increasing in $f(\theta^{(1)}, L)$, and relative inverse demand p^s_{12} is independent of $\theta^{(2)}$. Then, the maximizer $\theta^{(1)\star}$ is homogeneous of degree zero in (L_1, L_2), and since $p^l_{12} - p^s_{12}$ only depends on $\theta^{(1)\star}$, $p^l_{12} - p^s_{12}$ is homogeneous of degree zero in (L_1, L_2) as well.

If, in addition, g is independent of L_1 and L_2 – such that F can be written as $F(\theta, L) \equiv g(f(\theta^{(1)}, L), \theta^{(2)}, L_{-12})$ – p^l_{12} and p^s_{12} are homogeneous of degree zero in (L_1, L_2) individually.

To demonstrate the application of Theorem 1 and Conditions 1 and 2, consider the following specific example:

Example 1. A firm produces a single output with a constant elasticity of substitution (CES) production function in the (labor) inputs L_1 and L_2. The firm can choose the elasticity of substitution in the long run; greater flexibility in production (a higher elasticity of substitution) incurs greater costs. Formally, let F be given by

$$F(\theta, L) = (1 - C(\theta)) \left(\alpha L_1^\theta + (1-\alpha) L_2^\theta\right)^{\beta/\theta}$$

with $\alpha, \beta \in (0, 1)$, $C : \Theta \to (0, 1)$ a strictly increasing, convex C^2 cost function, and $\Theta = (0, 1)$. This means that the firm chooses an elasticity of substitution (equal to $1/(1-\theta)$) between 1 and ∞ in the long run. Assuming that C satisfies Inada conditions, the maximizer $\theta^\star$ will be interior, regular, and unique (because of strict concavity of F), such that Theorem 1 is applicable.

First, note that F is homogeneous in (L_1, L_2) such that (by Remark 1) Condition 2 is satisfied. Therefore, any increase in the input ratio L_1/L_2 induces the firm to adjust the elasticity of substitution in a way that raises relative inverse demand for L_1 versus L_2. Alternatively, following Corollary 1, relative inverse demand is less elastic in the long run than in the short run.

Second, suppose the production function has an additional, additive component:

$$F(\theta, L) = (1 - C(\theta)) \left(\alpha L_1^\theta + (1-\alpha) L_2^\theta\right)^{\beta/\theta} + G(L_1, L_2).$$

Now, Condition 1 is satisfied, but not Condition 2. So, again any increase in L_1/L_2 induces the firm to adjust the elasticity of substitution to raise relative inverse demand for L_1 versus L_2, but Corollary 1 does not apply, because the elasticity of relative inverse demand with respect to $l_{12} \equiv L_1/L_2$ may not be well-defined.

Third, suppose the cost for flexibility is represented by an additive, rather than multiplicative, cost function:

$$F(\theta, L) = \left(\alpha L_1^\theta + (1-\alpha) L_2^\theta\right)^{\beta/\theta} - C(\theta).$$

Now, neither of the two conditions is satisfied, and, indeed, the difference $p_{12}^l - p_{12}^s$ is not homogeneous of degree zero in (L_1, L_2). In particular, a proportional increase in L_1 and L_2 has a positive effect on the marginal product of θ and thus on $\theta^\star$. An increase in θ, in turn, raises relative inverse demand for the more heavily used of the two inputs. So, starting with $L_1 < L_2$ and increasing L_1/L_2 by a close to proportional increase in both inputs, the firm increases flexibility θ in the long run, which has a negative effect on relative inverse demand for the less heavily used input L_1 versus L_2.[17]

Yet, even in the case of an additive cost function, an increase in L_1/L_2 in direction of the isoquant always causes an adjustment of θ with a positive effect on relative inverse demand for L_1 versus L_2. For illustration, consider the first order condition for a maximum of F with respect to θ (which is necessary and sufficient due to concavity of F in θ):

$$-\frac{1}{\theta^2}\left(L_1^\theta + L_2^\theta\right)^{\frac{1}{\theta}} \log\left(L_1^\theta + L_2^\theta\right) + \frac{1}{\theta}\left(L_1^\theta \log L_1 + L_2^\theta \log L_2\right)\left(L_1^\theta + L_2^\theta\right)^{\frac{1-\theta}{\theta}} - D_\theta C(\theta) = 0.$$

Increasing L_1/L_2 in direction of the isoquant, only the term $\frac{1}{\theta}\left(L_1^\theta \log L_1 + L_2^\theta \log L_2\right)$ changes: the expression $\left(L_1^\theta + L_2^\theta\right)$ is unchanged by definition of the isoquant. Taking the directional derivative of $\frac{1}{\theta}\left(L_1^\theta \log L_1 + L_2^\theta \log L_2\right)$:

$$\begin{aligned} &D_{(L_1,L_2)=(1,d_I)}\left[\frac{1}{\theta}\left(L_1^\theta \log L_1 + L_2^\theta \log L_2\right)\right] \\ =&\frac{1}{\theta}\left(\theta \log L_1 + 1\right) L_1^{\theta-1} + \frac{d_I}{\theta}\left(\theta \log L_2 + 1\right) L_2^{\theta-1}, \end{aligned}$$

and with $d_I = -l_{12}^{\theta-1}$: $D_{(L_1,L_2)=(1,d_I)}\left[\frac{1}{\theta}\left(L_1^\theta \log L_1 + L_2^\theta \log L_2\right)\right] = L_1^{\theta-1} \log l_{12}$ (as in

[17] Acemoglu (2007) also discusses this example in his Example 2. He provides a numerical example in which an increase in L_1 causes an adjustment of θ with a negative effect on the relative marginal product of L_1 versus L_2.

Corollary 1, $l_{12} \equiv L_1/L_2$). It follows that $D_{(L_1,L_2)=(1,d_I)} D_\theta F(\theta, L) \gtreqless 0 \Leftrightarrow L_1 \gtreqless L_2$. Hence, the long-run value of θ increases in response to the change in (L_1, L_2) if and only if $L_1 > L_2$.[18] Relative inverse demand is given by $p^s_{12} = l_{12}^{\theta-1}$, which increases in θ if and only if $L_1 > L_2$. It follows that (i) if $L_1 > L_2$, $\theta^\star$ increases, which has a positive effect on relative inverse demand, and (ii) if $L_1 < L_2$, $\theta^\star$ decreases, which has a positive effect on relative inverse demand as well.

Proof of the Local Principle Finally, I prove Theorem 1. The idea of the proof is to choose a line in the (L_1, L_2)-plane and apply the original LeChatelier Principle on the line. It turns out that the results become informative about relative inverse demand exactly when the line is chosen tangent to the isoquant I_{12}. This leads to part 2 of the theorem; part 1 then follows quickly.

The key step is hence to prove part 2 of the theorem, which I repeat here for convenience.

Lemma (Part 2 of Theorem 1)**.** *Let F and Θ satisfy Assumptions 1 and 2. Let $\bar{L} \in \mathbb{R}^M_{++}$ and suppose that $\theta^\star(\bar{L}) \in \Theta^\circ$ is a regular and unique maximizer of $F(\theta, \bar{L})$ on Θ. Denote $\theta^\star(\bar{L})$ by $\bar{\theta}$, $D_{L_1} L_2(L_1; \bar{\theta}, \bar{L})$ by d_I and the directional derivative $v_1 D_{L_1} p^{s(l)}_{12}(\cdot) + v_2 D_{L_2} p^{s(l)}_{12}(\cdot)$ by $D_{(L_1,L_2)=(v_1,v_2)} p^{s(l)}_{12}(\cdot)$. Then:*

$$D_{(L_1,L_2)=(1,d_I)} p^s_{12}(\bar{\theta}, \bar{L}) \le D_{(L_1,L_2)=(1,d_I)} p^l_{12}(\bar{L})$$

with strict inequality if and only if $D_{(L_1,L_2)=(1,d_I)} \theta^\star(\bar{L}) \neq \mathbf{0}$.

Proof. Fix $\bar{L}$ and $\bar{\theta}$. Using that $p^l_{12}(L) \equiv p^s_{12}(\theta^\star(L), L)$:

$$\begin{aligned} & D_{(L_1,L_2)=(1,d_I)} p^l_{12}(\bar{L}) - D_{(L_1,L_2)=(1,d_I)} p^s_{12}(\bar{\theta}, \bar{L}) \\ = & \left[D_\theta p^s_{12}(\bar{\theta}, \bar{L}) \right] \left[D_{(L_1,L_2)=(1,d_I)} \theta^\star(\bar{L}) \right], \end{aligned}$$

where $D_{(L_1,L_2)=(1,d_I)} \theta^\star(\bar{L})$ is the N-dimensional vector collecting $D_{(L_1,L_2)=(1,d_I)} \theta^\star_i(\bar{L})$ for $i = 1, 2, ..., N$.

Computing the first vector of the product yields:

$$\begin{aligned} & D_\theta p^s_{12}(\bar{\theta}, \bar{L}) \\ = & \frac{1}{D_{L_2} F(\bar{\theta}, \bar{L})} \left[D^2_{L_1 \theta} F(\bar{\theta}, \bar{L}) - \frac{D_{L_1} F(\bar{\theta}, \bar{L})}{D_{L_2} F(\bar{\theta}, \bar{L})} D^2_{L_2 \theta} F(\bar{\theta}, \bar{L}) \right], \end{aligned}$$

[18] Since $D^2_{\theta\theta} F(\theta, L) < 0$ everywhere, the implicit function theorem applies: $D_{(L_1,L_2)=(1,d_I)} \theta^\star = -D_{(L_1,L_2)=(1,d_I)} \left[D_\theta F(\theta, L) \right] / \left[D^2_{\theta\theta} F(\theta, L) \right]$, which has the sign of $D_{(L_1,L_2)=(1,d_I)} \left[D_\theta F(\theta, L) \right]$.

and with $-D_{L_1}F(\bar{\theta},\bar{L})/D_{L_2}F(\bar{\theta},\bar{L}) = d_I$:

$$\begin{aligned} & D_\theta p_{12}^s(\bar{\theta},\bar{L}) \\ = & \frac{1}{D_{L_2}F(\bar{\theta},\bar{L})}\left[D^2_{L_1\theta}F(\bar{\theta},\bar{L}) + d_I D^2_{L_2\theta}F(\bar{\theta},\bar{L})\right]. \end{aligned}$$

For the second vector, note that $\theta^\star(\bar{L})$ satisfies the first order conditions

$$D_\theta F(\theta^\star(\bar{L}),\bar{L}) = 0,$$

and because $\theta^\star(\bar{L})$ is a regular maximizer, the implicit function theorem yields:

$$\begin{aligned} & D_{(L_1,L_2)=(1,d_I)}\theta^\star(\bar{L}) \\ = & -\left[D^2_{\theta\theta}F(\bar{\theta},\bar{L})\right]^{-1}\left[D^2_{\theta L_1}F(\bar{\theta},\bar{L}) + d_I D^2_{\theta L_2}F(\bar{\theta},\bar{L})\right]^T. \end{aligned}$$

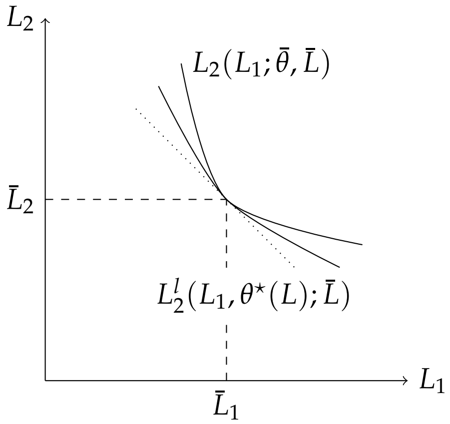
Then, combine the computed expressions to obtain:

$$\begin{aligned} & D_{L_1}p_{12}^l(\bar{L}) - D_{L_1}p_{12}^s(\bar{\theta},\bar{L}) \\ = & -\frac{1}{D_{L_2}F(\bar{\theta},\bar{L})}\left[D^2_{L_1\theta}F(\bar{\theta},\bar{L}) + d_I D^2_{L_2\theta}F(\bar{\theta},\bar{L})\right] \\ & \times\left[D^2_{\theta\theta}F(\bar{\theta},\bar{L})\right]^{-1}\left[D^2_{\theta L_1}F(\bar{\theta},\bar{L}) + d_I D^2_{\theta L_2}F(\bar{\theta},\bar{L})\right]^T \\ \geq & 0 \end{aligned}$$

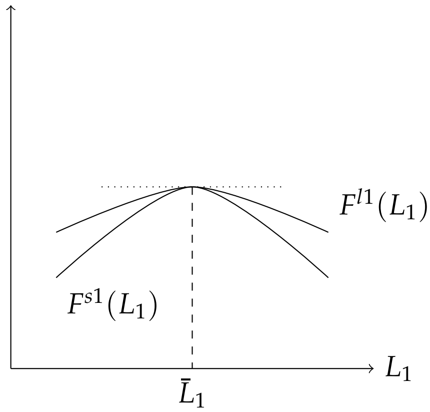
with strict inequality whenever $\left[D^2_{\theta L_1}F(\bar{\theta},\bar{L}) + d_I D^2_{\theta L_2}F(\bar{\theta},\bar{L})\right] \neq \mathbf{0}$, because $\theta^\star(\bar{L})$ is a regular maximizer and hence $\left[D^2_{\theta\theta}F(\bar{\theta},\bar{L})\right]^{-1}$ is negative definite. □

For part 1, note that any marginal change in (L_1, L_2) can be decomposed into a change along the isoquant and a proportionate change in both inputs. Formally, any direction vector $v \in \mathbb{R}^2$ can be decomposed into $v = s'(1, d_I) + s''(\bar{L}_1, \bar{L}_2)$, with $s', s'' \in \mathbb{R}$. Moreover, if $v \in \bar{V}$, meaning that direction v increases L_1/L_2 when starting from $(\bar{L}_1, \bar{L}_2)$, then s' is strictly positive. Decomposing the directional derivative

$$D_{(L_1,L_2)=v}\left[p_{12}^l(\bar{L}) - p_{12}^s(\bar{\theta},\bar{L})\right]$$



(a) Long-run and short-run isoquants at the level $F(\theta(\bar L),\bar L) = F(\bar\theta,\bar L)$. The short-run isoquant $L_2(L_1;\bar\theta,\bar L)$ is defined as in Definition 2. For the long-run isoquant replace $\bar\theta$ by $\theta^\star(L_1,L_2,\bar L_{-12})$ in the definition.



(b) Long-run and short-run production functions on the tangent line $L_2 = \bar L_2 + d_I(L_1 - \bar L_1)$ to the isoquant at level $F(\bar\theta,\bar L)$.

Figure 2

accordingly, we obtain:

$$\begin{aligned} &D_{(L_1,L_2)=v}\left[p^l_{12}(\bar L) - p^s_{12}(\bar\theta,\bar L)\right] \\ =&s' D_{(L_1,L_2)=(1,d_I)}\left[p^l_{12}(\bar L) - p^s_{12}(\bar\theta,\bar L)\right] + s'' D_{(L_1,L_2)=(\bar L_1,\bar L_2)}\left[p^l_{12}(\bar L) - p^s_{12}(\bar\theta,\bar L)\right]. \end{aligned}$$

If now $p^l_{12} - p^s_{12}$ is locally homogeneous of degree zero in (L_1,L_2), the second term of the sum is zero. Part 2 and $s' > 0$ then establish the if-part of part 1. Inversely, if $p^l_{12} - p^s_{12}$ is not locally homogeneous of degree zero in (L_1,L_2), then we can find a $v \in \bar V$ sufficiently close to $(\bar L_1,\bar L_2)$ or $-(\bar L_1,\bar L_2)$ such that s' is arbitrarily close to zero and s'' has the opposite sign of $D_{(L_1,L_2)=(\bar L_1,\bar L_2)}\left[p^l_{12}(\bar L) - p^s_{12}(\bar\theta,\bar L)\right]$. Then we have $D_{(L_1,L_2)=v}\left[p^l_{12}(\bar L) - p^s_{12}(\bar\theta,\bar L)\right] < 0$, completing the proof of part 1.

A simple way to understand the proof of Theorem 1 is the following. Restrict the production function to a line tangent to the isoquant at the point $(\bar L_1,\bar L_2)$. The tangent line is the dotted line in Figure 2a, which graphs short-run and long-run isoquants through $(\bar L_1,\bar L_2)$. Denote the restriction of the objective function to this line by $F^{s1}(L_1)$ when holding θ fixed at $\bar\theta$ (as in the short run), and by $F^{l1}(L_1)$ when letting θ adjust according to $\theta^\star(L)$ (as in the long run).[19] The two functions are graphed in Figure 2b.

[19]Formally, define $F^{s1}(L_1) \equiv F(\bar\theta, L_1, h(L_1), \bar L_{-12})$ and $F^{l1}(L_1) \equiv F(\theta(L_1,h(L_1),\bar L_{-12}), L_1, h(L_1), \bar L_{-12})$

The crucial step is to realize that the second-order derivatives of these functions inform about the changes in relative inverse demand in the short and the long run. To see this, consider a marginal increase in L_1 starting at $\bar{L}_1$: since this moves the pair (L_1, L_2) in direction of the isoquant, both F^{s1} and F^{l1} remain unchanged. Considering a marginal increase in L_1 starting at some point just above $\bar{L}_1$, a decrease in F^{s1} indicates that the marginal product of L_1 must have fallen relative to that of L_2 (compared to the marginal products at $(\bar{L}_1, \bar{L}_2)$ and holding θ constant); otherwise the effect of the increase in L_1 should still be sufficient to offset the negative effect of the accompanying decrease in L_2. The same holds for F^{l1}, only that θ is allowed to adjust when changing L_1 and L_2.

Finally, note that F^{l1} is an upper bound of F^{s1} with equality at $\bar{L}_1$ (see Figure 2b). This means that at points just above $\bar{L}_1$, the decrease in F^{s1} from marginally increasing L_1 must be greater than the decrease of F^{l1}. So, the relative marginal product of L_1 versus L_2 declines by more when holding θ constant (in the short run) than when allowing θ to adjust (in the long run).[20]

2.2. The Global Principle

The results of the preceding subsection are not entirely satisfactory for two reasons: they provide only local results, in the sense that they apply to infinitesimal changes in input quantities, and they require differentiability of the objective in θ (plus a suitable structure of Θ). In the present subsection, I employ the tools of monotone comparative statics[21] to derive a version of the LeChatelier Principle for Relative Demand that applies to discrete changes in input quantities of arbitrary size and does not require differentiability assumptions.[22] Importantly, this makes the LeChatelier Principle for Relative Demand applicable to problems with discrete choice spaces.

To apply monotone comparative statics, I assume:

Assumption 3. *The set Θ is a lattice.*

with h the tangent line to the isoquant: $h(L_1) = \bar{L}_2 + d_I(L_1 - \bar{L}_1)$.

[20] Formally, one can show that $D^2F^{s1}(\bar{L}_1) = D_{L_2}F(\bar{\theta}, \bar{L}) \left[D_{(L_1,L_2)=(1,d_I)} p^s_{12}(\bar{\theta}, \bar{L}) \right]$ and $D^2F^{l1}(\bar{L}_1) = D_{L_2}F(\bar{\theta}, \bar{L}) \left[D_{(L_1,L_2)=(1,d_I)} p^l_{12}(\bar{L}) \right]$. Then, the fact that F^{l1} is an upper bound of F^{s1} with equality at $\bar{L}_1$ implies that $D^2F^{l1}(\bar{L}_1) \geq D^2F^{s1}(\bar{L}_1)$ and hence: $D_{(L_1,L_2)=(1,d_I)} p^l_{12}(\bar{L}) \geq D_{(L_1,L_2)=(1,d_I)} p^s_{12}(\bar{\theta}, \bar{L})$. Note that this sketches an alternative to the formal proof of part 2 presented above. I do not use this alternative because it does not readily yield the qualification regarding strict inequality.

[21] See Appendix C for the monotone comparative statics results applied here. For an extensive collection of basic results in monotone comparative statics, see Topkis (1998).

[22] The procedure parallels the generalization of the original LeChatelier Principle by Milgrom and Roberts (1996).

Since the concepts used in monotone comparative statics are not as widely known in economics as the tools of differential calculus applied in the local principle, I briefly define the key concepts for deriving the global principle.[23] First, a real-valued function f defined on a lattice X is *quasisupermodular* if $f(x') \geq (>) f(x' \wedge x'') \Rightarrow f(x'') \leq (<) f(x' \vee x'')$, where $x' \wedge x''$ denotes the infimum of x' and x'', and $x' \vee x''$ denotes the supremum of x' and x''. Second, a real-valued function f defined on $X \times S$, with X and S partially ordered sets, has the *single crossing property* in $(x; s)$ if for $x' > x''$ and $s' > s''$, $f(x', s'') \geq (>) f(x'', s'') \Rightarrow f(x', s') \geq (>) f(x'', s')$.

With these definitions and Assumption 3, I can state the global version of the LeChatelier Principle for Relative Demand:

Theorem 2. *Let F and Θ satisfy Assumptions 1 and 3 and let F be quasisupermodular in θ. Moreover, let $\bar{L} \in \mathbb{R}^M_{++}$, denote $\theta^\star(\bar{L})$ by $\bar{\theta}$,[24] and let F have the single crossing property in $(\theta; L_1)$ and in $(L_1; \theta)$ on $\Theta \times I_{12}(\bar{\theta}, \bar{L}) \times \{\bar{L}_{-12}\}$, meaning that for $\theta' > \theta''$ and $L_1' > L_1''$:*[25]

$$
\begin{aligned}
&F(\theta', L_1'', L_2(L_1''; \bar{\theta}, \bar{L}), L_{-12}) \geq (>) F(\theta'', L_1'', L_2(L_1''; \bar{\theta}, \bar{L}), L_{-12}) \\
&\Rightarrow F(\theta', L_1', L_2(L_1'; \bar{\theta}, \bar{L}), L_{-12}) \geq (>) F(\theta'', L_1', L_2(L_1'; \bar{\theta}, \bar{L}), L_{-12}) \\
\textit{and } &F(\theta'', L_1', L_2(L_1'; \bar{\theta}, \bar{L}), L_{-12}) \geq (>) F(\theta'', L_1'', L_2(L_1''; \bar{\theta}, \bar{L}), L_{-12}) \\
&\Rightarrow F(\theta', L_1', L_2(L_1'; \bar{\theta}, \bar{L}), L_{-12}) \geq (>) F(\theta', L_1'', L_2(L_1''; \bar{\theta}, \bar{L}), L_{-12}).
\end{aligned}
$$

Then:

1. *For any $\bar{\bar{L}}_1, \bar{\bar{L}}_2 \in \mathbb{R}^2_{++}$ with $\bar{\bar{L}}_1 / \bar{\bar{L}}_2 \geq \bar{L}_1 / \bar{L}_2$:*

$$p^s_{12}(\bar{\theta}, \bar{\bar{L}}_1, \bar{\bar{L}}_2, \bar{L}_{-12}) \leq p^l_{12}(\bar{\bar{L}}_1, \bar{\bar{L}}_2, \bar{L}_{-12})$$

if $p^l_{12}(L_1, L_2, \bar{L}_{-12}) - p^s_{12}(\bar{\theta}, L_1, L_2, \bar{L}_{-12})$ is homogeneous of degree zero in (L_1, L_2).

2. *For any $(\bar{\bar{L}}_1, \bar{\bar{L}}_2) \in I_{12}(\bar{\theta}, \bar{L})$ with $\bar{\bar{L}}_1 / \bar{\bar{L}}_2 \geq \bar{L}_1 / \bar{L}_2$:*

$$p^s_{12}(\bar{\theta}, \bar{\bar{L}}_1, \bar{\bar{L}}_2, \bar{L}_{-12}) \leq p^l_{12}(\bar{\bar{L}}_1, \bar{\bar{L}}_2, \bar{L}_{-12}).$$

Proof. First, I prove part 2. By Theorem 4 in Milgrom and Shannon (1994) (see Appendix C), quasisupermodularity of F in θ and the single crossing property in $(\theta; L_1)$

[23] I use the definitions from Milgrom and Shannon (1994).

[24] $\text{argmax}_{\theta \in \Theta} F(\theta, L)$ is a sublattice of Θ by Corollary 2 in Milgrom and Shannon (1994), and compact and non-empty by Assumption 1. Birkhoff (1967) shows that a compact sublattice (in the interval topology of Frink, 1942) is complete, such that $\theta^\star(\bar{L})$ exists in $\text{argmax}_{\theta \in \Theta} F(\theta, L)$.

[25] Note that this is the standard definition of the single crossing property as given in the text above, only that the isoquant equation $L_2(L_1; \tilde{\theta}, \tilde{L})$ is substituted for L_2 into F.

on the isoquant $I_{12}(\bar{\theta}, \bar{L})$ imply that $\bar{\theta} \leq \theta^\star(\bar{\bar{L}}_1, \bar{\bar{L}}_2, \bar{L}_{-12})$. To simplify the notation, denote $\theta^\star(\bar{\bar{L}}_1, \bar{\bar{L}}_2, \bar{L}_{-12})$ by $\bar{\bar{\theta}}$. Moreover, for any $L'_1 > L''_1$, $F(\bar{\theta}, L'_1, L_2(L'_1; \bar{\theta}, \bar{L}), \bar{L}_{-12}) = F(\bar{\theta}, L''_1, L_2(L''_1; \bar{\theta}, \bar{L}), \bar{L}_{-12})$ (by definition of the isoquant).

Then, the single crossing property in $(L_1; \theta)$ implies $F(\bar{\bar{\theta}}, L'_1, L_2(L'_1; \bar{\theta}, \bar{L}), \bar{L}_{-12}) \geq F(\bar{\bar{\theta}}, L''_1, L_2(L''_1; \bar{\theta}, \bar{L}), \bar{L}_{-12})$, and hence

$$D_{L_1} F(\bar{\bar{\theta}}, \bar{\bar{L}}_1, \bar{\bar{L}}_2, \bar{L}_{-12}) + \bar{d}_I D_{L_2} F(\bar{\bar{\theta}}, \bar{\bar{L}}_1, \bar{\bar{L}}_2, \bar{L}_{-12}) \geq 0,$$

where $\bar{d}_I = D_{L_1} L_2(\bar{\bar{L}}_1; \bar{\theta}, \bar{L}) = -p^s_{12}(\bar{\theta}, \bar{\bar{L}}_1, \bar{\bar{L}}_2, \bar{L}_{-12})$. This implies

$$\begin{aligned} p^l_{12}(\bar{\bar{L}}_1, \bar{\bar{L}}_2, \bar{L}_{-12}) &= \frac{D_{L_1} F(\bar{\bar{\theta}}, \bar{\bar{L}}_1, \bar{\bar{L}}_2, \bar{L}_{-12})}{D_{L_2} F(\bar{\bar{\theta}}, \bar{\bar{L}}_1, \bar{\bar{L}}_2, \bar{L}_{-12})} \\ &\geq p^s_{12}(\bar{\theta}, \bar{\bar{L}}_1, \bar{\bar{L}}_2, \bar{L}_{-12}). \end{aligned}$$

For part 1 – with $\bar{\bar{L}}_1, \bar{\bar{L}}_2$ now an arbitrary pair in $\mathbb{R}^2_{++}$ satisfying $\bar{\bar{L}}_1/\bar{\bar{L}}_2 \geq \bar{L}_1/\bar{L}_2$ – choose a pair $(L^{aux}_1, L^{aux}_2) \in I_{12}(\bar{\theta}, \bar{L})$ such that $L^{aux}_1/L^{aux}_2 = \bar{\bar{L}}_1/\bar{\bar{L}}_2$. Then, part 2 says that $p^l_{12}(L^{aux}_1, L^{aux}_2, \bar{L}_{-12}) - p^s_{12}(\bar{\theta}, L^{aux}_1, L^{aux}_2, \bar{L}_{-12}) \geq 0$. But zero homogeneity of $p^l_{12} - p^s_{12}$ in (L_1, L_2) implies that $p^l_{12}(L^{aux}_1, L^{aux}_2, \bar{L}_{-12}) - p^s_{12}(\bar{\theta}, L^{aux}_1, L^{aux}_2, \bar{L}_{-12}) = p^l_{12}(\bar{\bar{L}}_1, \bar{\bar{L}}_2, \bar{L}_{-12}) - p^s_{12}(\bar{\theta}, \bar{\bar{L}}_1, \bar{\bar{L}}_2, \bar{L}_{-12})$, which establishes the desired result. □

Remark 3. The condition that F has the single crossing property in $(\theta; L_1)$ on the isoquant can be replaced by the somewhat weaker condition that $F(\theta, L''_1, L_2(L''_1; \bar{\theta}, \bar{L}), \bar{L}_{-12})$ dominates $F(\theta, L'_1, L_2(L'_1; \bar{\theta}, \bar{L}), \bar{L}_{-12})$ in the interval dominance order, proposed by Quah and Strulovici (2009), for any L'_1, L''_1 with $L'_1 < L''_1$.

The interval dominance order requires the single crossing property to hold only for closed intervals in Θ on which $F(\theta, L'_1, L_2(L'_1; \bar{\theta}, \bar{L}), \bar{L}_{-12})$ is maximized at the supremum. Put differently, for any closed interval on which $F(\theta, L'_1, L_2(L'_1; \bar{\theta}, \bar{L}), \bar{L}_{-12})$ attains its maximum at the supremum of the interval, also $F(\theta, L''_1, L_2(L''_1; \bar{\theta}, \bar{L}), \bar{L}_{-12})$ must attain its maximum at the supremum. See Appendix C for a general definition.

Lemma 4 in Appendix C implies that, if F obeys the interval dominance order and is quasisupermodular in θ, then the maximizer $\theta^\star$ is increasing in L_1 on the isoquant $I_{12}(\bar{\theta}, \bar{L})$. The rest of the proof of Theorem 2 goes through unchanged.

The theorem translates the local LeChatelier Principle for Relative Demand, Theorem 1, into a global version.[26] It says that, if $p^l_{12} - p^s_{12}$ is homogeneous of degree zero

[26]More precisely, Theorem 2 is a generalization of Theorem 1: the assumptions of quasisupermodularity and the single crossing properties are always satisfied locally (after appropriate relabeling of θ and components thereof) when F is C^2 in (θ, L) and Θ is a smooth manifold.

in (L_1, L_2), any increase in the relative quantity L_1/L_2 has a positive effect on $p^l_{12} - p^s_{12}$; that is, any increase in L_1/L_2 induces an adjustment of long-run inputs that increases relative inverse demand for L_1 versus L_2. If $p^l_{12} - p^s_{12}$ is not homogeneous of degree zero in (L_1, L_2), the result continues to hold along the isoquant with respect to L_1 and L_2.

Intuitively, the local LeChatelier Principle for Relative Demand relies on the symmetry of local complementarity relations: an increase in L_1 along the isoquant with respect to L_1 and L_2 is complementary to θ if and only if an increase in θ is more complementary to L_1 than to L_2 in the sense that it has a positive effect on the relative marginal product of L_1 versus L_2. This local symmetry of complementarity relations is implicit in the assumption that F is C^2 in (θ, L).

Analogously, the global principle requires symmetry of global complementarity relations. Quasisupermodularity and single crossing properties are a direct formalization of this requirement: single crossing in $(\theta; L_1)$ on the isoquant ensures that an increase in L_1 along the isoquant is complementary to θ; single crossing in $(L_1; \theta)$ on the isoquant ensures that an increase in θ is more complementary to L_1 than to L_2; and quasisupermodularity may be understood as a weak form of complementarity within θ – or between the different components of θ if such exist – which ensures that the maximizers $\theta^\star(L_1, L_2(L_1; \bar{\theta}, \bar{L}), \bar{L}_{-12})$ are ordered for any two values of L_1, although the set Θ may not be totally ordered. It follows that an increase in L_1 along the isoquant always induces an increase in θ, and the increase in θ always has a positive effect on relative inverse demand for L_1 versus L_2.

To apply the global principle, characterizations of quasisupermodularity and single crossing in terms of more familiar concepts may prove helpful. Such characterizations are given by Milgrom and Shannon (1994).[27] More generally, by imposing not more than the comparably weak lattice structure on Θ and by dispensing with differentiability assumptions in θ, the theorem is applicable to problems with discrete choice spaces. It thereby covers many natural settings, such as a firm's problem of choosing between different organizational forms or of choosing between different technologies with no natural measure of distance between them.

[27] Characterizations of the interval dominance order in terms of more familiar concepts are given by Quah and Strulovici (2009).

3. Implications for Directed Technical Change

I apply the LeChatelier Principle for Relative Demand to advance the theory of directed technical change. In particular, I derive general results for the effect of directed technical change on relative factor prices, summarized in four theorems about weak and strong relative equilibrium bias (defined below), both for a class of static and a class of dynamic models.

While the dynamic models require several assumptions for a regular asymptotic behavior of the economy, which restrict the generality of the results on directed technical change without being essential for any of them, the static models allow to focus on essential assumptions and thus to exploit the full generality of the LeChatelier Principle for Relative Demand.

I discuss applications of the developed theory in subsection 3.3.

3.1. Static Models

The class of static models consists of all models with M (non-technology) production inputs, a set of feasible technologies Θ, and an (aggregate production) function $F : \Theta \times \mathbb{R}^M_{++} \to \mathbb{R}$ for which we can define:[28]

(i) a *short-run equilibrium* such that, for a fixed technology $\theta \in \Theta$ and a given aggregate supply of production inputs $L \in \mathbb{R}^M_{++}$, the prices of production inputs are given by $p = D_L F(\theta, L)$;

(ii) a *long-run equilibrium* such that, for a given aggregate supply of production inputs $L \in \mathbb{R}^M_{++}$, the prices of production inputs are given by $p = D_L F(\theta^\star(L), L)$ and technology is determined as $\theta^\star(L) = \sup \arg\max_{\theta \in \Theta} F(\theta, L)$.

Acemoglu (2007) presents four specific examples of models in which we can naturally define short-run and long-run equilibria that satisfy the listed requirements. All of these four models feature a unique final good, produced by a continuum of identical firms, and M production inputs, all traded on perfectly competitive markets and supplied inelastically by a continuum of consumers. The models differ in the rules that determine the technology used in long-run equilibrium, with different implica-

[28] Of course, it is possible to define such equilibria in virtually any model. While this is formally sufficient to apply the results derived in the present subsection, such an application clearly makes sense only if the required definitions of short-run and long-run equilibrium coincide with some natural or conventional definition of equilibrium in the given model.

tions for the curvature of aggregate production F at a point of equilibrium. I sketch the four models in Appendix A.

For later reference, I summarize the crucial requirement for application of the LeChatelier Principle for Relative Demand in the following assumption:

Assumption 4. *For a fixed technology $\theta \in \Theta$ and a given aggregate supply of production inputs $L \in \mathbb{R}^M_{++}$, short-run equilibrium input prices are:*

$$p_i^{s\star}(\theta, L) \equiv D_{L_i} F(\theta, L)$$

for $i = 1, 2, ..., M$.

For a given aggregate supply of production inputs $L \in \mathbb{R}^M_{++}$, long-run equilibrium input prices are:

$$p_i^{l\star}(L) \equiv D_{L_i} F(\theta^\star(L), L)$$

for $i = 1, 2, ..., M$, where $\theta^\star(L) \equiv \sup \operatorname{argmax}_{\theta \in \Theta} F(\theta, L)$.

The goal is to derive general conditions for weak and strong relative equilibrium bias in models satisfying Assumption 4. Following the literature (see, for example Acemoglu 2007), I define:[29]

Definition 3. There is *weak relative equilibrium bias* with respect to inputs 1 and 2 at input supply $\bar{L} \in \mathbb{R}^M_{++}$ if

$$D_{(L_1, L_2) = v} \frac{p_1^{s\star}(\bar{\theta}, \bar{L})}{p_2^{s\star}(\bar{\theta}, \bar{L})} \leq D_{(L_1, L_2) = v} \frac{p_1^{l\star}(\bar{L})}{p_2^{l\star}(\bar{L})} \quad \text{for all } v \in \left\{(v_1, v_2) \in \mathbb{R}^2 | v_1/\bar{L}_1 > v_2/\bar{L}_2\right\}$$

where $\bar{\theta} = \sup \operatorname{argmax}_{\theta \in \Theta} F(\theta, \bar{L})$.

[29] Acemoglu (2007) defines weak relative equilibrium bias as

$$D_{L_1} \frac{p_1^{s\star}(\bar{\theta}, \bar{L})}{p_2^{s\star}(\bar{\theta}, \bar{L})} \leq D_{L_1} \frac{p_1^{l\star}(\bar{L})}{p_2^{l\star}(\bar{L})}$$

and strong relative equilibrium bias as

$$D_{L_1} \frac{p_1^{l\star}(\bar{L})}{p_2^{l\star}(\bar{L})} > 0.$$

This is convenient when analyzing settings in which F is homothetic in (L_1, L_2), and $\theta^\star(L)$ is homogeneous of degree zero in (L_1, L_2). Under these conditions, the inequalities either hold for all directions v that increase L_1/L_2, or for none, such that Acemoglu's definitions are equivalent to Definitions 3 and 4. Since I do not impose such restrictions on F, I use the more detailed Definitions 3 and 4.

Definition 4. There is *strong relative equilibrium bias* with respect to inputs 1 and 2 at input supply $\bar{L} \in \mathbb{R}^M_{++}$ if

$$D_{(L_1,L_2)=v} \frac{p_1^{l\star}(\bar{L})}{p_2^{l\star}(\bar{L})} > 0 \quad \text{for all } v \in \left\{(v_1, v_2) \in \mathbb{R}^2 | v_1/\bar{L}_1 > v_2/\bar{L}_2\right\}.$$

In words: there is weak relative equilibrium bias if any infinitesimal increase in the relative supply L_1/L_2 induces technical change biased towards L_1 versus L_2. If the effect of the induced technical change is sufficient to make the relative price of L_1 versus L_2 increase in long-run equilibrium, there is strong relative equilibrium bias. I will also say that there is *weak* or *strong relative equilibrium bias along the isoquant* whenever the corresponding inequality holds for the direction $v = (1, d_I)$ with $d_I = -D_{L_1}F(\bar{\theta}, \bar{L})/D_{L_2}F(\bar{\theta}, \bar{L})$.

Weak Relative Equilibrium Bias Since the relative input prices in Assumption 4 are identical to relative inverse demand functions, the main result on weak relative equilibrium bias follows as a direct corollary from the LeChatelier Principle for Relative Demand.

Corollary 2. *Consider any model in which factor prices satisfy Assumption 4 and let F and Θ satisfy Assumptions 1 and 2. Let $\bar{L} \in \mathbb{R}^M_{++}$ and suppose that $\theta^\star(\bar{L}) \in \Theta^\circ$ is a regular and unique maximizer of $F(\theta, \bar{L})$ on Θ. Denote $\theta^\star(\bar{L})$ by $\bar{\theta}$ and $d_I = -D_{L_1}F(\bar{\theta}, \bar{L})/D_{L_2}F(\bar{\theta}, \bar{L})$. Then:*

1. *There is weak relative equilibrium bias at $\bar{L}$ if and only if $p_1^{l\star}/p_2^{l\star} - p_1^{s\star}/p_2^{s\star}$ is locally homogeneous of degree zero in (L_1, L_2) at $\bar{L}$. Formally:*

$$D_{(L_1,L_2)=v} \frac{p_1^{s\star}(\bar{\theta}, \bar{L})}{p_2^{s\star}(\bar{\theta}, \bar{L})} \leq D_{(L_1,L_2)=v} \frac{p_1^{l\star}(\bar{L})}{p_2^{l\star}(\bar{L})} \quad \textit{for all } v \in \left\{(v_1, v_2) \in \mathbb{R}^2 | v_1/\bar{L}_1 > v_2/\bar{L}_2\right\}$$

(with strict inequality if and only if $D_{(L_1,L_2)=(1,d_I)}\theta^\star(\bar{L}) \neq \mathbf{0}$)
if and only if $D_{(L_1,L_2)=(\bar{L}_1,\bar{L}_2)} \left[p_1^{l\star}(\bar{L})/p_2^{l\star}(\bar{L}) - p_1^{s\star}(\bar{\theta}, \bar{L})/p_2^{s\star}(\bar{\theta}, \bar{L})\right] = 0$.

2. *There is always weak relative equilibrium bias along the isoquant at $\bar{L}$. Formally:*

$$D_{(L_1,L_2)=(1,d_I)} \frac{p_1^{s\star}(\bar{\theta}, \bar{L})}{p_2^{s\star}(\bar{\theta}, \bar{L})} \leq D_{(L_1,L_2)=(1,d_I)} \frac{p_1^{l\star}(\bar{L})}{p_2^{l\star}(\bar{L})}$$

(with strict inequality if and only if $D_{(L_1,L_2)=(1,d_I)}\theta^\star(\bar{L}) \neq \mathbf{0}$).

Remark 4. Corollary 2 applies to infinitesimal changes in input supply only, as it is derived from the local LeChatelier Principle for Relative Demand. A global version of the corollary follows directly from the global LeChatelier Principle for Relative Demand. The global version gives conditions for a global version of weak relative equilibrium bias, by which any discrete increase in relative input supply L_1/L_2 induces technical change biased towards L_1 versus L_2. As the global LeChatelier Principle for Relative Demand, it applies to settings with discrete choice spaces and non-differentiable production functions.

The corollary says that there is weak relative equilibrium bias, whenever a proportionate increase in the two inputs under consideration does not induce biased technical change.[30] If a proportionate increase in supply does induce biased technical change, there is still weak relative equilibrium bias along a range of directions, including the direction of the isoquant $(1, d_I)$. There is no role for factor-augmenting technologies in the corollary.

The results make substantial progress in the theory of directed technical change. First, they suggest that weak relative equilibrium bias is prevalent in a wide class of endogenous technology models, independently of technologies being purely factor-augmenting. Second, they provide precise information about the limitations of the phenomenon. The applications discussed in Section 3.3 give an impression of the practical relevance of this generalization.

Strong Relative Equilibrium Bias The only existing condition for strong relative equilibrium bias with respect to inputs L_1 and L_2 is that the elasticity of substitution between the two inputs be greater than some (finite) threshold level (Acemoglu 1998, Kiley 1999, and Acemoglu 2007). The condition lacks generality: first, it is only proved for settings with factor-augmenting technologies and homothetic production functions; second, one can easily show that a more general condition cannot take the form of a lower threshold for the elasticity of substitution. An example in which there is strong relative equilibrium bias if and only if the elasticity of substitution is below, rather than above, a threshold level is the following:

Example 2. Let the production function be given by

$$F(\theta, L) = \left[\theta_1^{2-3\rho} L_1^\rho + \theta_2^{2-3\rho} L_2^\rho\right]^{1/\rho} - c_1\theta_1 - c_2\theta_2$$

[30]Note that the operationalizations (in terms of conditions on F) of the condition that the difference between long-run and short-run relative prices be homogeneous of degree zero in (L_1, L_2) from Section 2.1 are valid here as well. In particular, the sufficient Conditions 1 and 2 apply.

with $\rho \in [1/2-\epsilon, 1/2+\epsilon]$ and ϵ small enough such that first order conditions determine $\theta^\star(L) = \operatorname{argmax}_{\theta \in \mathbb{R}^2_+} F(\theta, L)$. Then, in short-run equilibrium:

$$\frac{p_1^{s\star}}{p_2^{s\star}} = \left(\frac{\theta_1}{\theta_2}\right)^{2-3\rho} \left(\frac{L_1}{L_2}\right)^{\rho-1}.$$

First order conditions for $\theta^\star$ imply:

$$\frac{\theta_1^\star}{\theta_2^\star} = \left(\frac{c_1}{c_2}\right)^{\frac{1}{1-3\rho}} \left(\frac{L_1}{L_2}\right)^{\frac{\rho}{3\rho-1}}.$$

Combining the two impressions to derive long-run equilibrium prices yields:

$$\frac{p_1^{l\star}}{p_2^{l\star}} = \chi \left(\frac{L_1}{L_2}\right)^{\frac{1-2\rho}{3\rho-1}}$$

with χ some parametric expression. Finally, note that the elasticity of substitution between L_1 and L_2 is $1/(1-\rho)$.

Then, for an elasticity of substitution closely above (below) 2, the relative price decreases (increases) in relative supply in long-run equilibrium. Put differently, there is strong relative equilibrium bias if and only if the elasticity of substitution between L_1 and L_2 is below the threshold of 2.

The following theorem gives a general condition for strong relative equilibrium bias. The proof is delegated to the paragraph Proof of Theorem 3 at the end of the present subsection.

Theorem 3. *Consider any model in which factor prices satisfy Assumption 4 and let F and Θ satisfy Assumptions 1 and 2. Let $\bar{L} \in \mathbb{R}^M_{++}$ and suppose that $\theta^\star(\bar{L}) \in \Theta^\circ$ is a regular and unique maximizer of $F(\theta, \bar{L})$ on Θ. Denote $\theta^\star(\bar{L})$ by $\bar{\theta}$ and $d_I = -D_{L_1}F(\bar{\theta}, \bar{L})/D_{L_2}F(\bar{\theta}, \bar{L})$. Then:*

1. *If $p_1^{l\star}/p_2^{l\star}$ is locally homogeneous of degree zero in (L_1, L_2) at $\bar{L}$, then there is strong relative equilibrium bias if and only if the Hessian of F with respect to (θ, L_1, L_2) is not negative semi definite on the isoquant with respect to (L_1, L_2).*
   Formally: If $D_{(L_1,L_2)=(\bar{L}_1,\bar{L}_2)}\left[p_1^{l\star}(\bar{L})/p_2^{l\star}(\bar{L})\right] = 0$, then:

$$D_{(L_1,L_2)=v}\frac{p_1^{l\star}(\bar{L})}{p_2^{l\star}(\bar{L})} > 0 \;\forall v \in \left\{(v_1, v_2) \in \mathbb{R}^2 | v_1/\bar{L}_1 > v_2/\bar{L}_2\right\}$$
$$\Leftrightarrow \exists h \in \mathbb{R}^{N+2} : -d_I h_{N+1} + h_{N+2} = 0 \wedge h^T D^2_{(\theta,L_1,L_2)(\theta,L_1,L_2)} F(\bar{\theta}, \bar{L}) h > 0.$$

2. *There is strong relative equilibrium bias along the isoquant if and only if the Hessian of F with respect to (θ, L_1, L_2) is not negative semi definite on the isoquant with respect to (L_1, L_2). Formally:*

$$D_{(L_1,L_2)=(1,d_I)} \frac{p_1^{l\star}(\bar{L})}{p_2^{l\star}(\bar{L})} > 0$$
$$\Leftrightarrow \exists h \in \mathbb{R}^{N+2} : -d_I h_{N+1} + h_{N+2} = 0 \wedge h^T D^2_{(\theta,L_1,L_2)(\theta,L_1,L_2)} F(\bar{\theta}, \bar{L}) h > 0.$$

Remark 5. Theorem 3 applies to infinitesimal changes in input supply only, but it directly implies conditions for a global version of strong relative equilibrium bias, by which the relative price in long-run equilibrium increases in response to discrete increases in relative supply of arbitrary size. Apply the fundamental theorem of calculus to compute the change in the relative price in response to a discrete change in relative supply,

$$\frac{p_1^{l\star}(\bar{\bar{L}}_1, \bar{\bar{L}}_2, \bar{L}_{-12})}{p_2^{l\star}(\bar{\bar{L}}_1, \bar{\bar{L}}_2, \bar{L}_{-12})} - \frac{p_1^{l\star}(\bar{L})}{p_2^{l\star}(\bar{L})} = \int_0^1 D_{(L_1,L_2)=v} \frac{p_1^{l\star}(\bar{L}_1 + v_1 x, \bar{L}_2 + v_2 x, \bar{L}_{-12})}{p_2^{l\star}(\bar{L}_1 + v_1 x, \bar{L}_2 + v_2 x, \bar{L}_{-12})} \, dx$$

with $v_1 = \bar{\bar{L}}_1 - \bar{L}_1$ and $v_2 = \bar{\bar{L}}_2 - \bar{L}_2$, and note that, for $\bar{\bar{L}}_1 / \bar{\bar{L}}_2 > \bar{L}_1 / \bar{L}_2$, the change will be positive whenever the necessary and sufficient conditions for local strong relative equilibrium bias are satisfied at every point on the line between $(\bar{\bar{L}}_1, \bar{\bar{L}}_2, \bar{L}_{-12})$ and $\bar{L}$.

Note that the conditions for strong relative equilibrium bias may well be satisfied in general equilibrium models satisfying Assumption 4: if different sets of firms decide about technology θ and non-technology inputs L, there is no reason for F to be jointly concave in θ and L.[31] Indeed, in three of the four models sketched in Appendix A, strong relative equilibrium bias is possible. Only in Economy D, final good firms, which demand the non-technology inputs, choose their technology themselves, such that their production functions, identical to aggregate production F, must be locally concave in (θ, L) at a point of long-run equilibrium; strong relative equilibrium bias is therefore impossible in Economy D.

When technologies are purely factor-augmenting, meaning that F takes the form $G(\theta_1 L_1, \theta_2 L_2) - C(\theta)$, and F is homothetic, the existing condition (see, for example, Theorem 1 in Acemoglu 2007) that the elasticity of substitution between L_1 and L_2 exceeds a certain threshold level is clearly equivalent to the condition from Theorem

[31] Also, note the similarity to the necessary and sufficient condition for strong absolute equilibrium bias of technology from Theorem 4 in Acemoglu 2007.

3, that the Hessian of F is not negative semi definite at a point of long-run equilibrium. The new condition, being much more generally applicable, suggests that strong relative equilibrium bias arises under fairly weak conditions in a wide class of models, independently of factor-augmenting technologies.[32]

Proof of Theorem 3 The key step in the proof of Theorem 3 is to prove part 2 of the theorem, which I repeat here for convenience.

Lemma (Part 2 of Theorem 3)**.** *Consider any model in which factor prices satisfy Assumption 4 and let F and Θ satisfy Assumptions 1 and 2. Let $\bar{L} \in \mathbb{R}^M_{++}$ and suppose that $\theta^\star(\bar{L}) \in \Theta^\circ$ is a regular and unique maximizer of $F(\theta, \bar{L})$ on Θ. Denote $\theta^\star(\bar{L})$ by $\bar{\theta}$ and $d_I = -D_{L_1}F(\bar{\theta}, \bar{L})/D_{L_2}F(\bar{\theta}, \bar{L})$.*

Then, there is strong relative equilibrium bias along the isoquant if and only if the Hessian of F with respect to (θ, L_1, L_2) is not negative semi definite on the isoquant with respect to (L_1, L_2). Formally:

$$\begin{aligned} & D_{(L_1,L_2)=(1,d_I)} \frac{p_1^{l\star}(\bar{L})}{p_2^{l\star}(\bar{L})} > 0 \\ \Leftrightarrow & \exists h \in \mathbb{R}^{N+2} : -d_I h_{N+1} + h_{N+2} = 0 \wedge h^T D^2_{(\theta,L_1,L_2)(\theta,L_1,L_2)} F(\bar{\theta}, \bar{L}) h > 0. \end{aligned}$$

Proof. $(\Rightarrow)$ The proof uses that

$$D_\theta F(\theta^\star(L), L) \equiv 0 \text{ in some neighborhood of } \bar{L}, \tag{3.1}$$

and

$$\left[\left(D_{(L_1,L_2)=(1,d_I)} \theta^\star(\bar{L}) \right)^T, 1, d_I \right] \left[D^2_{\theta(\theta,L_1,L_2)} F(\bar{\theta}, \bar{L}) \right] = \mathbf{0}, \tag{3.2}$$

where (3.1) follows from $\theta^\star$ being a unique interior maximizer of F at $\bar{L}$ and (3.2) follows from taking the derivative $D_{(L_1,L_2)=(1,d_I)} [D_\theta F(\theta^\star(L), L)]$ at $\bar{L}$.

Define the long-run production function $F^l(L) \equiv F(\theta^\star(L), L)$ as in Section 2.1, fix L_{-12} at $\bar{L}_{-12}$ and use the linear function $h(L_1) = \bar{L}_2 + d_I(L_1 - \bar{L}_1)$ to define $F^{l1}(L_1) \equiv F^l(L_1, h(L_1), \bar{L}_{-12})$.

[32]For application of part 1 of Theorem 3, note that the operationalizations (in terms of conditions on F) of the condition that the long-run relative price be homogeneous of degree zero in (L_1, L_2) from Section 2.1 are valid here as well. In particular, the sufficient Condition 2 applies.

The second derivative of F^{l1} at $\bar{L}_1$ is equal in sign to $D_{(L_1,L_2)=(1,d_I)}\left(p_1^{l\star}(\bar{L})/p_2^{l\star}(\bar{L})\right)$:

$$\begin{aligned}
& D^2 F^{l1}(\bar{L}_1) \\
= \; & D_{(L_1,L_2)=(1,d_I)}\Big(D_{L_1}F(\theta^\star(\bar{L}),\bar{L}) + d_I D_{L_2}F(\theta^\star(\bar{L}),\bar{L}) + D_\theta F(\theta^\star(\bar{L}),\bar{L}) \\
& \times \left[D_{(L_1,L_2)=(1,d_I)}\theta^\star(\bar{L}) \right] \Big) \\
= \; & D_{(L_1,L_2)=(1,d_I)} \left[D_{L_1}F(\theta^\star(\bar{L}),\bar{L}) + d_I D_{L_2}F(\theta^\star(\bar{L}),\bar{L}) \right] \quad \text{(by (3.1))} \\
= \; & D_{(L_1,L_2)=(1,d_I)} \left[D_{L_1}F(\theta^\star(\bar{L}),\bar{L}) \right] + d_I D_{(L_1,L_2)=(1,d_I)} \left[D_{L_2}F(\theta^\star(\bar{L}),\bar{L}) \right] \\
= \; & D_{L_2}F(\theta^\star(\bar{L}),\bar{L}) \left[D_{(L_1,L_2)=(1,d_I)} \frac{p_1^{l\star}(\bar{L})}{p_2^{l\star}(\bar{L})} \right].
\end{aligned}$$

Starting with the second equality, $D^2F^{l1}(\bar{L}_1)$ can be written as follows:

$$\begin{aligned}
D^2 F^{l1}(\bar{L}_1) &= D_{(L_1,L_2)=(1,d_I)} \left[D_{(L_1,L_2)} F(\theta^\star(\bar{L}),\bar{L})\,(1,d_I)^T \right] \\
&= \left(\left[D_{(L_1,L_2)=(1,d_I)}\theta^\star(\bar{L}) \right]^T, 1, d_I \right) \left[D^2_{(L_1,L_2)(\theta,L_1,L_2)} F(\theta^\star(\bar{L}),\bar{L})\,(1,d_I)^T \right],
\end{aligned}$$

where $D^2_{(L_1,L_2)(\theta,L_1,L_2)}F(\bar{\theta},\bar{L})$ are columns $N+1$ and $N+2$ of the Hessian of F with respect to (θ, L_1, L_2). Finally, using (3.2), adding the first N columns of the Hessian does not alter the result:

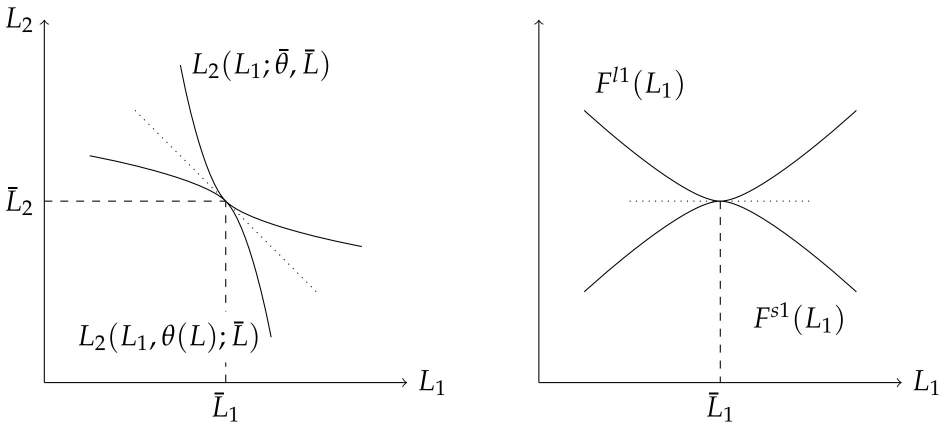
$$\begin{aligned}
D^2 F^{l1}(\bar{L}_1) = \; & \left(\left[D_{(L_1,L_2)=(1,d_I)}\theta^\star(\bar{L}) \right]^T, 1, d_I \right) \left[D^2_{(\theta,L_1,L_2)(\theta,L_1,L_2)} F(\theta^\star(\bar{L}),\bar{L}) \right] \\
& \times \left(\left[D_{(L_1,L_2)=(1,d_I)}\theta^\star(\bar{L}) \right]^T, 1, d_I \right)^T.
\end{aligned}$$

It follows that if $D_{(L_1,L_2)=(1,d_I)}\left(p_1^{l\star}(\bar{L})/p_2^{l\star}(\bar{L})\right) > 0$, then $\left[\left(D_{(L_1,L_2)=(1,d_I)}\theta^\star(\bar{L}) \right)^T 1, d_I \right]^T$ satisfies the properties of vector h in the theorem.

($\Rightarrow$) By definition of $\theta^\star$, the function $F^{l1}(L_1)$ is an upper envelope to the family of functions

$$\left\{ F^w(L_1) \equiv F(\bar{\theta} + w(L_1 - \bar{L}_1), L_1, \bar{L}_2 + d_I(L_1 - \bar{L}_1), \bar{L}_{-12}) | w \in \mathbb{R}^N \right\}.$$

Since $F^w(\bar{L}_1) = F^{l1}(\bar{L}_1)$ for any w, the second derivative of F^{l1} must be greater at $\bar{L}_1$



(a) Long-run and short-run isoquants at the level $F(\theta(\bar L),\bar L) = F(\bar\theta,\bar L)$. The figure repeats Figure 2a for the case that the long-run isoquant is non-convex.

(b) Long-run and short-run production functions on the tangent line $L_2 = \bar L_2 + d_I(L_1 - \bar L_1)$ to the isoquant at level $F(\bar\theta,\bar L)$. The figure repeats Figure 2b for the case that the long-run production function is non-concave.

Figure 3

than that of F^w:

$$\begin{aligned} D^2F^{l1}(\bar L_1) &\geq D^2F^w(\bar L_1) \\ &= \left(w^T,1,d_I\right)\left[D^2_{(\theta,L_1,L_2)(\theta,L_1,L_2)}F(\bar\theta,\bar L)\right]\left(w^T,1,d_I\right)^T \end{aligned}$$

for any w. So, if there exists a vector $h = (w^T,1,d_I)^T \in \mathbb{R}^{N+2}$ as required in the theorem, then $D^2F^{l1}(\bar L_1) > 0$ and hence $D_{(L_1,L_2)=(1,d_I)}\left(p_1^{l\star}(\bar L)/p_2^{l\star}(\bar L)\right) > 0$. □

For part 1, decompose any given vector $v \in \left\{(v_1,v_2) \in \mathbb{R}^2 | v_1/\bar L_1 > v_2/\bar L_2\right\}$ into isoquant and scaling component, $s'(1,d_I)$ with $s' > 0$ and $s''(\bar L_1,\bar L_2)$, as in the proof of the local LeChatelier Principle for Relative Demand. With $D_{(L_1,L_2)=(\bar L_1,\bar L_2)}\left(p_1^{l\star}(\bar L)/p_2^{l\star}(\bar L)\right) = 0$ we obtain

$$D_{(L_1,L_2)=v}\left(p_1^{l\star}(\bar L)/p_2^{l\star}(\bar L)\right) = s'D_{(L_1,L_2)=(1,d_I)}\left(p_1^{l\star}(\bar L)/p_2^{l\star}(\bar L)\right),$$

and part 2 establishes part 1 of the theorem.

Figure 3 illustrates the proof of part 2. It repeats Figure 2 for the case that the long-run production function F^l is not concave along the tangent line to the isoquant at $(\bar L_1,\bar L_2)$ (the dotted line in Figure 3a). The proof of part 2 shows that the curvature of the long-run production function F^{l1} in Figure 3b determines the sign of $D_{(L_1,L_2)=(1,d_I)}\left(p_1^{l\star}(\bar{L})/p_2^{l\star}(\bar{L})\right)$.

To understand this intuitively, consider an infinitesimal increase in L_1/L_2 in direction of the isoquant. If F^{l1} is concave at $\bar{L}_1$, its slope increases with the infinitesimal increase in L_1/L_2. But this means, that another infinitesimal move along the isoquant towards L_1 will have a positive effect on long-run output (since the effect at $\bar{L}_1$ is zero); hence, the marginal product of L_1 must have increased relative to that of L_2. So, if and only if the slope of F^{l1} strictly increases (locally) in L_1, there is strong relative equilibrium bias. This is the case in Figure 3b but not in Figure 2b. Since F^{l1} is an upper envelope to all functions $F(\tilde{\theta}(L), L_1, h(L_1), \bar{L}_{-12})$ with $\tilde{\theta}(L_1)$ some function of L_1 with $\tilde{\theta}(\bar{L}_1) = \bar{\theta}$, the condition on the curvature of F^{l1} translates into a condition on the curvature of F in (θ, L_1, L_2).

3.2. Dynamic Models

Dynamic models give an explicit description of the adjustment of technology in response to a change in input supply. The short-run response of input prices to such a change is then the contemporaneous change of input prices at the time t at which the input supply change occurs. The long-run response is the cumulative change in input prices after the economy has converged to its new steady state.

With these associations, we can transfer the results for static models to any dynamic model with M (non-technology) production inputs, a (technology) variable $\theta \in \mathbb{R}^N_{++}$, an (aggregate production) function $F : \mathbb{R}^N_+ \times \mathbb{R}^M_{++} \to \mathbb{R}$, $(\theta, L) \mapsto F(\theta, L)$ that is homogeneous in θ, and a set $S \subset \mathbb{R}^N_{++}$ for which we can define:

(i) an equilibrium such that, for any given aggregate supply of production inputs $L \in \mathbb{R}^M_{++}$, any technology $\theta_t \in \mathbb{R}^N_+$, and at any time t, the prices of production inputs are given by $D_L F(\theta_t, L)$;

(ii) a balanced growth path such that, for any given aggregate supply of production inputs $L \in \mathbb{R}^M_{++}$, technology on the balanced growth path $\theta_t^{bgp}(L)$ is a multiple of $\theta^\star(L) \equiv \sup \operatorname{argmax}_{\theta \in S} F(\theta, L)$ for any t.

For later reference, I summarize:

Assumption 5. *The function $F : \mathbb{R}^N_+ \times \mathbb{R}^M_{++} \to \mathbb{R}$, $(\theta, L) \mapsto F(\theta, L)$ is homogeneous in θ. For any aggregate supply of production inputs $L \in \mathbb{R}^M_{++}$, any technology $\theta_t \in \mathbb{R}^N_+$, and at* *any time t, equilibrium input prices are:*

$$p_i(\theta_t, L) \equiv D_{L_i} F(\theta_t, L)$$

for $i = 1, 2, ..., M$.

For any aggregate supply of production inputs $L \in \mathbb{R}^M_{++}$, technology on a balanced growth path $\theta_t^{bgp}(L)$ is a multiple of $\theta^\star(L) \equiv \sup \operatorname{argmax}_{\theta \in S \subset \mathbb{R}^N_+} F(\theta, L)$, that is: for any t, there exists a scalar s_t such that $s_t \theta^\star(L) = \theta_t^{bgp}(L)$.

In Appendix B, I show that a generalized version of the standard dynamic model of directed technical change, as developed by Acemoglu (1998) and Kiley (1999), satisfies Assumption 5. The generalization concerns the set of feasible technologies: while the standard model only allows for factor-augmenting technologies within a CES production function with two input factors, the generalized version in Appendix B only imposes a set of weak regularity conditions on the production function F, which guarantee existence of a unique balanced growth path (see Assumption 6 in Appendix B). In particular, I assume F to be linear homogeneous and strictly quasiconcave in technology $\theta \in \mathbb{R}^N_+$ (for existence of a unique constant growth path), linear homogeneous in all rival inputs[33] (for microeconomic consistency), and to satisfy an Inada type condition in technology (for balancedness of the constant growth path).

Technology variables in the dynamic model are aggregates of different varieties of intermediate inputs, each of which embodies a certain technology and is supplied by a monopolist.[34] Progress in technology i is the result of deliberate research investment into technology i and either improves the quality of intermediate inputs or increases the number of available varieties aggregated to θ_i. In any case, investment into technology i increases the variable θ_i. The non-technology inputs L are supplied inelastically on competitive factor markets such that factor prices equal marginal products in equilibrium; hence, the first part of Assumption 5 is satisfied.

Moreover, Appendix B establishes the following lemma:

Lemma 2. *In the model of Appendix B, there exists a unique balanced growth path $\theta_t^{bgp}(L)$ for any aggregate input supply $L \in \mathbb{R}^M_{++}$.*

Let $\theta^\star(L) \equiv \sup \operatorname{argmax}_{\theta \in S^N_{\tilde{\alpha}}} F(\theta, L)$ with $S^N_{\tilde{\alpha}} = \left\{\theta \in \mathbb{R}^N_+ | \sum_{i=1}^N (1/\tilde{\alpha}_i)\theta_i = 1\right\}$. Then, for any $t > 0$, there exists a scalar $s_t > 0$ such that $\theta_t^{bgp}(L) = s_t \theta^\star(L)$.

[33] Rival inputs are intermediate goods embodying technology and non-technology inputs.

[34] This follows the endogenous growth models based on monopolistic competition, pioneered by Romer (1990), Grossman and Helpman (1991), and Aghion and Howitt (1992).

Thus, the model also satisfies the second part of Assumption 5.

To apply the results from the static setting, define the relative price

$$p_{12}(\theta_t, L) \equiv p_1(\theta_t, L)/p_2(\theta_t, L)$$

and the relative price on the balanced growth path

$$p_{12}^{bgp}(L) \equiv p_1(\theta_t^{bgp}(L), L)/p_2(\theta_t^{bgp}(L), L).$$

Starting from the balanced growth path at $\bar{L}$, the derivative $D_{(L_1,L_2)=v}p_{12}(\bar{\theta}_t^{bgp}, \bar{L})$ with $\bar{\theta}_t^{bgp} = \theta_t^{bgp}(\bar{L})$ is the contemporaneous change of the relative price in response to a change in (L_1, L_2) in direction v. The derivative $D_{(L_1,L_2)=v}p_{12}^{bgp}(\bar{L})$ is the corresponding long-run change in the relative price. For the former, technology is fixed at the old balanced growth path, for the latter it is adjusted to the new one.

The definitions of weak and strong relative equilibrium bias transfer to the dynamic setting in the natural way: there is weak relative equilibrium bias whenever the contemporaneous change of the relative price p_1/p_2 in response to any infinitesimal increase in relative supply L_1/L_2 is smaller than the long-run change, and there is strong relative equilibrium bias whenever the long-run change is strictly positive.

Corollary 3. *Consider any model that satisfies Assumption 5 and let F satisfy Assumptions 1 and 2. Let $\bar{L} \in \mathbb{R}^M_{++}$ and suppose that $\theta^\star(\bar{L}) \in S^\circ$ is a regular and unique maximizer of $F(\theta, \bar{L})$ on S. Denote $\theta_t^{bgp}(\bar{L})$ by $\bar{\theta}_t^{bgp}$ and $d_I = -D_{L_1}F(\bar{\theta}_t^{bgp}, \bar{L})/D_{L_2}F(\bar{\theta}_t^{bgp}, \bar{L})$. Then:*

1. *There is weak relative equilibrium bias at $\bar{L}$ if and only if $p_{12}^{bgp} - p_{12}$ is locally homogeneous of degree zero in (L_1, L_2) at $\bar{L}$. Formally:*

   $$D_{(L_1,L_2)=v}p_{12}(\bar{\theta}_t^{bgp}, \bar{L}) \leq D_{(L_1,L_2)=v}p_{12}^{bgp}(\bar{L}) \quad \text{for all } v \in \{(v_1, v_2) \in \mathbb{R}^2 | v_1/\bar{L}_1 > v_2/\bar{L}_2\}$$

   (with strict inequality if and only if $D_{(L_1,L_2)=(1,d_I)}\theta_t^{bgp}(\bar{L}) \neq \mathbf{0}$)
   if and only if $D_{(L_1,L_2)=(\bar{L}_1,\bar{L}_2)}\left[p_{12}^{bgp}(\bar{L}) - p_{12}(\bar{\theta}_t^{bgp})\right] = 0$.

2. *There is always weak relative equilibrium bias along the isoquant at $\bar{L}$. Formally:*

   $$D_{(L_1,L_2)=(1,d_I)}p_{12}(\bar{\theta}_t^{bgp}, \bar{L}) \leq D_{(L_1,L_2)=(1,d_I)}p_{12}^{bgp}(\bar{L})$$

   (with strict inequality if and only if $D_{(L_1,L_2)=(1,d_I)}\theta_t^{bgp}(\bar{L}) \neq \mathbf{0}$).

Corollary 4. *Consider any model in which factor prices satisfy Assumption 5 and let F satisfy Assumptions 1 and 2. Let $\bar{L} \in \mathbb{R}^M_{++}$ and suppose that $\theta^\star(\bar{L}) \in S^\circ$ is a regu-*

lar and unique maximizer of $F(\theta, \bar{L})$ *on* S. *Denote* $\theta_t^{bgp}(\bar{L})$ *by* $\bar{\theta}_t^{bgp}$, $\theta^\star(\bar{L})$ *by* $\bar{\theta}$, *and* $d_I = -D_{L_1}F(\bar{\theta}_t^{bgp}, \bar{L})/D_{L_2}F(\bar{\theta}_t^{bgp}, \bar{L})$. *Then:*

1. *If* p_{12}^{bgp} *is locally homogeneous of degree zero in* (L_1, L_2) *at* $\bar{L}$, *then there is strong relative equilibrium bias if and only if the Hessian of* F *with respect to* (θ, L_1, L_2) *is not negative semi definite on the isoquant with respect to* (L_1, L_2).
Formally: If $D_{(L_1,L_2)=(\bar{L}_1,\bar{L}_2)}p_{12}^{bgp}(\bar{L}) = 0$, *then:*

$$\begin{aligned} &D_{(L_1,L_2)=v}p_{12}^{bgp}(\bar{L}) > 0\, \forall v \in \{(v_1, v_2) \in \mathbb{R}^2 | v_1/\bar{L}_1 > v_2/\bar{L}_2\} \\ \Leftrightarrow &\exists h \in \mathbb{R}^{N+2} : -d_I h_{N+1} + h_{N+2} = 0 \wedge h^T D^2_{(\theta,L_1,L_2)(\theta,L_1,L_2)}F(\bar{\theta}, \bar{L})h > 0. \end{aligned}$$

2. *There is strong relative equilibrium bias along the isoquant if and only if the Hessian of* F *with respect to* (θ, L_1, L_2) *is not negative semi definite on the isoquant with respect to* (L_1, L_2). *Formally:*

$$\begin{aligned} &D_{(L_1,L_2)=(1,d_I)}p_{12}^{bgp}(\bar{L}) > 0 \\ \Leftrightarrow &\exists h \in \mathbb{R}^{N+2} : -d_I h_{N+1} + h_{N+2} = 0 \wedge h^T D^2_{(\theta,L_1,L_2)(\theta,L_1,L_2)}F(\bar{\theta}, \bar{L})h > 0. \end{aligned}$$

For the proof, observe that $p_{12}(\theta_t^{bgp}(L), L) = p_{12}(\theta^\star(L), L)$ for any L by Assumption 5. Then, the corollaries follow directly from Corollary 2 and Theorem 3.

Remark 6. In the model of Appendix B, the maximizer $\theta^\star(L)$ can never be interior, because the set S_α^N has no interior (see Lemma 2). Yet, the results of Corollaries 3 and 4 apply to the model.

To see that, eliminate θ_N from the maximization problem $\max_{\theta \in S_\alpha^N} F(\theta, L)$ to obtain an unconstrained problem with solution $\tilde{\theta}^\star(L)$. Use this solution and $\theta_N = \tilde{\alpha}_N - \sum_{i=1}^{N-1}(\tilde{\alpha}_N/\tilde{\alpha}_i)\tilde{\theta}_i^\star$ to represent input prices on the balanced growth path. Since $\tilde{\theta}^\star$ is always interior due to the Inada type condition in Assumption 6, Corollary 2 and Theorem 3 apply to the new representation of input prices.

In combination with the model in Appendix B, Corollaries 3 and 4 are a substantial generalization of existing results: as already suggested by the results for static models, factor-augmenting technologies are by no means essential for weak relative equilibrium bias, even in dynamic models with fully specified microfoundation. The curvature condition for strong equilibrium bias is valid in the dynamic setting as well.

3.3. Applications

The developed theory is relevant to three types of models in applied work. It makes the theory of relative equilibrium bias applicable to, first, models in which the functional form of aggregate production is endogenous and has no explicit solution, and second, to models with specific technologies that cannot be represented in factor-augmenting form. Third, it lends support to models that assume purely factor-augmenting technologies by proving that the assumption can be relaxed.

In the following, I demonstrate how the developed theory may be applied with gain to each of these settings.

Applications without explicit production structure The developed theory is applicable when an explicit description of the aggregate production function in terms of elementary functions does not exist. This is the case in models in which the allocation of production factors to different positions in the production process is endogenous and has no closed form solution. Assignment models, in which the allocation of workers to tasks or sectors is endogenous, are an important example.

Assignment models have recently been applied to study (i) the effect of (exogenous) technical change on the occupational employment distribution (for example, Autor and Dorn 2013); (ii) the effect of international trade on the occupational employment and wage distribution (for example,Costinot and Vogel 2010); (iii) optimal tax schedules with endogenous occupation choices (for example, Rothschild and Scheuer 2013). I sketch two examples of how to apply the general results on relative equilibrium bias in assignment models.

First, consider a simple model with skilled labor supply L_1, unskilled labor supply L_2, and production function $\tilde{F}(\theta, L_1, L_2^m, L_2^r)$ where L_2^m denotes the quantity employed of manual labor, L_2^r the quantity of routine labor.[35] The allocation of unskilled workers to manual and routine labor is endogenous: unskilled workers choose between supplying one unit of manual labor or γ units of routine labor, where γ is distributed with density g on an interval $\Gamma \in \mathbb{R}_{++}$ among unskilled workers.

In an interior solution of the assignment problem, all unskilled workers with $\gamma \geq$

[35] See Autor and Dorn (2013) for a similar model with exogenous technical change. The example is motivated by the (empirically supported) hypothesis that, over the last two decades, computer technologies have primarily replaced workers performing routine tasks, while complementing workers performing abstract cognitive tasks, typically performed by skilled workers, and manual tasks (Autor et al. 2003). The hypothesis potentially explains the observations of employment and wage polarization on the labor markets of many industrialized economies (Autor et al. 2006; Goos and Manning 2007).

$\bar{\gamma} \in \Gamma$ supply routine labor; so, aggregate production becomes

$$\tilde{F}(\theta, L_1, L_2 \int_{\gamma<\bar{\gamma}} dG, L_2 \int_{\gamma \geq \bar{\gamma}} \gamma \, dG).$$

Since $\bar{\gamma}$ depends on L_1 and L_2, an explicit solution for output as a function of θ, L_1, and L_2 is typically not available, even if we know the functional forms of $\tilde{F}$ and G. Moreover, even if technologies are factor-augmenting in $\tilde{F}$ – that is, $\tilde{F}$ takes the form $f(\theta_1 L_2, \theta_2 L_2^m, \theta_3 L_2^r) - c(\theta)$ – computing the equilibrium technology[36] and its response to changes in L_1 and L_2 directly is potentially involved. Also, existing results for factor-augmenting technologies are not applicable as technology does not augment L_1 and L_2 in aggregate production.

Yet, given that an aggregate production function $F(\theta, L_1, L_2)$ exists, we can apply the results on relative equilibrium bias from Section 3.1. In particular, if the (weak) conditions for weak relative equilibrium bias are satisfied, an increase in the relative skill supply L_1/L_2 induces technical change that raises the relative marginal product of skilled versus unskilled workers, $D_{L_1}F/D_{L_2}F$. Since the marginal products in F are easily shown to equal the corresponding partial derivatives of $\tilde{F}$, $D_{L_i}F = D_{L_i}\tilde{F}$ for $i = 1, 2$,[37] they correspond to the wage of skilled and the average wage of unskilled workers. If, in addition, technical change that complements skilled workers is biased towards manual versus routine labor, this simple model is capable of endogenously generating technical change that raises the skill premium (defined as skilled wages over the average of unskilled wages) and creates wage and employment polarization at the same time.[38]

As a second example, consider a standard Heckscher-Ohlin model with transportation (or communication) costs of θ per unit of international trade. The two factors in the model are again skilled and unskilled workers. Denote their supply by L_1^X and L_2^X for country $X = A, B$. The two goods are h and l with h the skill-intensive good of the two. Denote their prices by p_h^X and p_l^X for country $X = A, B$. Finally, let A be

[36] I assume that the equilibrium technology in the example is determined as in the static models analyzed in Section 3.1.

[37] The reason is that the marginal effect of $\bar{\gamma}$ on $\tilde{F}$ is zero: $\bar{\gamma}$ is determined such that

$$D_{L_2^m}\tilde{F}(\theta, L_1, L_2^m, L_2^r)L_2 = D_{L_2^r}\tilde{F}(\theta, L_1, L_2^m, L_2^r)L_2\bar{\gamma};$$

hence:

$$D_{\bar{\gamma}}\tilde{F}(\theta, L_1, L_2^m, L_2^r) = D_{L_2^m}\tilde{F}(\theta, L_1, L_2^m, L_2^r)L_2 - D_{L_2^r}\tilde{F}(\theta, L_1, L_2^m, L_2^r)L_2\bar{\gamma} = 0.$$

[38] In this context, employment polarization means that skilled and manual employment increase relative to routine employment. Wage polarization means that wages of skilled and manual workers increase relative to wages of routine workers.

skill-abundant, that is $L_1^A/L_2^A > L_1^B/L_2^B$.

Then, A will export h and import l in equilibrium. With incomplete specialization, relative goods prices must satisfy $(p_h^A+\theta)/(p_l^A-\theta) = p_h^B/p_l^B$. Normalizing p_l^A to one, aggregate production in A is $p_h^A f_h(L_1^{A,h}, L_2^{A,h}) + f_l(L_1^{A,l}, L_2^{A,l}) - C(\theta)$, where f_j are sectoral production functions and C is a strictly decreasing cost function for reductions in transportation costs (improvements in transportation technology). Even with explicit functional forms for sectoral production, an explicit solution for aggregate production will typically not exist, due to, first, the endogenous assignment of workers to sectors, and second, the endogenous determination of the relative price. So again, existing results for factor-augmenting technologies are not applicable, even if technology is purely factor-augmenting in the sectoral production functions. Also, direct computation of the relationship between labor supply and transportation cost is potentially involved.

Yet, we know by standard arguments that a decrease in transportation costs increases the skill premium in country A. Thus, whenever the (weak) conditions for weak relative equilibrium bias from Corollary 2 are satisfied, an increase in the relative skill supply in A induces an endogenous reduction in transportation costs, with a positive effect on the skill premium in A. The simple model hence offers a new perspective on the relationship between inequality and international trade: while existing research has focused on how exogenous expansions in the volume of international trade affect inequality, both directly (for example, Costinot and Vogel 2010) and via induced technical change (for example, Acemoglu 2003b), the model suggests to view reductions in transportation costs as a skill-biased technical change themselves; the implication, enabled by the theory developed in the preceding sections, is that a surge in the relative skill supply in advanced economies increases incentives to reduce transportation costs, expanding international trade and thus inflating the skill premium.[39]

Applications with explicitly not factor-augmenting technologies The developed theory clearly applies to models with technologies of a specific, but not factor-augmenting form. Models with labor-replacing technologies, as in Zeira (1998), Acemoglu (2010), and more recently in studies of task automation (Hemous and Olsen 2014, Acemoglu and Restrepo 2016, among others), are an important example. Given the recently

[39] At first inspection, this seems to be in line with aggregate data: while the relative supply of skills increased remarkably during the second half of the 20th century in many industrialized economies (when stagnating in parts of the developing world), the volume of international trade began to increase with some delay in the 1980s and 1990s, supposedly contributing to the rise in wage inequality observed in many industrialized countries.

sparked interest in task automation,[40] we may see a surge of such models with the potential for application of the developed theory in the near future.

As a simple example, consider a model with skilled labor supply L_1, unskilled labor supply L_2, and production function $F(\theta, L_1, L_2) = f(\theta_1 L_1, \theta_2 L_2 + \theta_3) - C(\theta)$. Technology is not purely factor-augmenting, but may instead directly replace unskilled labor, so existing results on relative equilibrium bias do not apply.

Corollary 2, in contrast, is revealing: it turns out that weak relative equilibrium bias is unlikely in the presence of labor-replacing technologies. In particular, a proportionate increase in L_1 and L_2 is likely to induce biased technical change: it increases ceteris paribus the return to θ_2 relative to θ_3, while an increase in θ_2 relative to θ_3 is likely to decrease the skill premium.[41] If, for example, $C(\theta)$ takes the form $C(\theta) = \sum_{i=1}^{3} c_i \theta_i^2$ (and f is homothetic), we can easily verify that a proportionate increase in L_1 and L_2 induces technical change biased against skilled workers.[42] Intuitively, the more labor is available, the less attractive are technologies that dispense with the use of labor, and if these technologies mainly perform tasks of unskilled workers, their removal benefits the unskilled more than the skilled. This result implies that, when labor-replacing technologies are sufficiently important in the aggregate, an increase in the relative supply of skilled workers that comes close to a proportionate increase in both types of labor induces technical change biased towards the unskilled, as it discourages automation of unskilled worker tasks. The opposite conclusion clearly applies to a decrease in the relative supply of skilled workers that comes close to a proportionate decrease in both types of labor. In general, the results about the limitations of weak relative equilibrium bias in Section 3.1 show that weak relative equilibrium bias, and therewith also strong relative equilibrium bias, is unlikely in economies with a relevant role for

[40] The interest in task automation stems from predictions that new technologies will be able to replace workers, both skilled and unskilled, in a substantial share of occupations over the next two decades (Frey and Osborne 2013).

[41] Note that for a non-trivial function f, F satisfies neither Condition 1 nor Condition 2, which are sufficient for weak relative equilibrium bias.

[42] For $C(\theta) = \sum_{i=1}^{3} c_i \theta_i^2$, $\theta_2^\star$ and $\theta_3^\star$ must always satisfy: $\theta_3^\star = (c_2 \theta_2^\star)/(L_2 c_3)$. The first order conditions for θ_1 and θ_2 imply:

$$MRS\left(\frac{\theta_1 L_1}{\theta_2 L_2 + \theta_3}\right)\frac{L_1}{L_2} - \frac{\theta_1}{\theta_2} = 0$$

with $MRS(\cdot)$ some decreasing function (assuming f homothetic and concave). Substituting for θ_3 yields:

$$MRS\left(\frac{\theta_1 L_1}{\theta_2 (L_2 + c_2/L_2 c_3)}\right)\frac{L_1}{L_2} - \frac{\theta_1}{\theta_2} = 0.$$

Since the left hand side decreases both with a proportionate increase in L_1 and L_2 and in θ_1/θ_2, a proportionate increase in L_1 and L_2 decreases $\theta_1^\star/\theta_2^\star$. Therefore, the ratio $\theta_1^\star L_1/(\theta_2^\star L_2 + \theta_3^\star)$ must increase in the new equilibrium (relative to the old). In combination, these two changes imply that $D_{L_1} F$ must fall relative to $D_{L_2} F$.

labor-replacing technologies.

Applications with factor-augmenting technologies Finally, the developed theory applies to models in which assuming technologies to be purely factor-augmenting is feasible. First, it lends credibility to existing applications that assume technology to be purely factor-augmenting by proving that the assumption can be relaxed substantially. Second, it may motivate new applications in which the assumption of factor-augmenting technologies is feasible but appears an undue restriction of generality.

As an example for such a new application, consider the simple optimal taxation framework analyzed by Stiglitz (1987). There are skilled and unskilled workers and an aggregate production function with endogenous technology (potentially but not necessarily taking a factor-augmenting form). With exogenous technology, we know that the optimal utilitarian tax schedule involves a positive marginal tax rate for low incomes and a negative marginal tax rate for high incomes (Stiglitz 1987).

When technology is endogenous, Theorem 3 provides conditions for strong relative equilibrium bias. If these conditions are satisfied, the tax designer may want to induce unskilled workers to increase and skilled workers to decrease their labor supply: the resulting reduction in the relative supply of skilled workers induces technical change sufficiently biased against skilled workers to cause an overall decrease in the skill premium, reducing inequality. A negative marginal tax for low income earners and a positive marginal tax for high income earners may thus become optimal.

4. Conclusion

In this paper, I first presented the LeChatelier Principle for Relative Demand, which is a relevant extension to the theory of demand. It says that, whenever the relative inverse demand curve for two inputs is a function of the relative quantity of the two inputs rather than of the levels of the two inputs individually, it is less elastic (in the relative quantity) in the long run than in the short run. The basic version of the principle applies to infinitesimal changes in input quantities only and is restricted in generality by differentiability assumptions. The generalized version, relying on the theory of monotone comparative statics, applies to discrete changes in input quantities in non-differentiable optimization problems. It covers problems in which the only reasonable mathematical representation of the choice space is a discrete space, for example because there is no natural measure of distance between the alternatives (the choice between organizational forms or between discrete production technologies, among others).

Second, I applied the LeChatelier Principle for Relative Demand to propose a general theory of relative equilibrium bias. In particular, in a wide class of general equilibrium models with endogenous technology, there is weak relative equilibrium bias if and only if a proportionate increase in the two inputs under consideration does not induce biased technical change. Restricting input changes to isoquants, there is always weak relative equilibrium bias. The results show that weak relative equilibrium bias is prevalent in a fairly wide class of static models. Moreover, there is strong relative equilibrium bias if and only if the Hessian of the aggregate production function is not negative semi definite on the isoquant. None of the results requires existence of factor-augmenting technologies.

The general conditions for relative equilibrium bias transfer to a class of dynamic general equilibrium models with endogenous technology. Importantly, this class of models includes a generalized version of the standard dynamic model of directed technical change by Acemoglu (1998) and Kiley (1999). The generalized version, presented in detail in Appendix B, only imposes restrictions on aggregate production that are essential for existence of a unique balanced growth path. In particular, technologies may not be factor-augmenting (as they are in the standard model).

The developed theory has three main implications for applied work. First, the results in many existing applications of relative equilibrium bias are independent of the restrictive assumption of purely factor-augmenting technologies; the assumption can be relaxed substantially. Second, the general results open the theory of directed technical change to many potential future applications, especially to those incompatible with purely factor-augmenting technology. Such applications are of two types: applications in which the allocation of inputs to positions in the production process is endogenous and has no explicit solution, and applications with technologies of a specific but not (purely) factor-augmenting form. Examples for the former are assignment models of the labor market and Ricardian models of international trade; examples for the latter are models of labor-replacing technologies and models with transportation or communication technologies. I discussed potential applications within these examples in Section 3.3. Third, by providing necessary and sufficient conditions, the developed theory gives precise information about the limits of relative equilibrium bias; it thus clarifies where relative equilibrium bias cannot be applied. In particular, weak (and strong) relative equilibrium bias fail whenever a proportionate increase in the supply of the two inputs under consideration induces biased technical change.

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A. Static Models of Directed Technical Change: Four Examples

I sketch Economies D, M, C, and O from Acemoglu (2007). All of these four are examples of models that satisfy Assumption 4; the results on relative equilibrium bias from Section 3.1 therefore apply.

Economy D Economy D features a continuum of identical consumption good producers of measure one and a continuum of consumers. The consumption good producers produce a unique consumption good according to the production function $F(\theta^i, L^i)$, where $\theta^i \in \Theta$ denotes producer i's technology and $L^i \in \mathbb{R}^M_{++}$ the quantities of M (non-technology) production inputs employed by producer i. Each producer chooses its own technology from the set of feasible technologies Θ. Consumers are endowed with aggregate quantities $L \in \mathbb{R}^M_{++}$ of the M production inputs, which they supply inelastically on perfectly competitive markets, have monotone preferences over the consumption good, and own the consumption good producers.

For any fixed technology $\theta \in \Theta$ and aggregate input supply $L \in \mathbb{R}^M_{++}$, a short-run equilibrium consists of (i) a price vector p for the M production inputs,[43] and (ii) input quantities L^i for each consumption good producer i, such that all input markets clear, and consumption good producers maximize profits taking technology θ as given. It follows that, for continuously differentiable F, input prices must satisfy $p = D_L F(\theta, L)$ in any symmetric short-run equilibrium.

For a given aggregate input supply $L \in \mathbb{R}^M_{++}$, a long-run equilibrium consists of (i) an input price vector p, (ii) input quantities L^i, and (iii) technologies $\theta^i \in \Theta$, such that all input markets clear and consumption good producers maximize profits. Again for continuously differentiable F, prices must satisfy $p = D_L F(\theta^i, L)$ and technology $\theta^i \in \operatorname{argmax}_{\theta \in \Theta} F(\theta, L)$ in any symmetric long-run equilibrium.

Importantly, F must be jointly concave in θ and L at any point of long-run equilibrium, because consumption good producers choose both technology and non-technology inputs to maximize profits. Hence, by Theorem 3, there can never be strong relative equilibrium bias in this economy.

Economy C Economy C is as Economy D, only that technology is chosen by a public agent to maximize aggregate production. We can define the short-run equilibrium as in Economy D, with the same implication for input prices that $p = D_L F(\theta, L)$.

[43] Normalize the price of the consumption good to one.

The long-run equilibrium now consists of (i) an input price vector p, (ii) input quantities L^i, and (iii) a technology $\theta^\star \in \Theta$, such that all input markets clear, consumption good producers maximize profits, and technology $\theta^\star$ maximizes aggregate output. Prices again satisfy $p = D_L F(\theta^\star, L)$ and technology $\theta^\star \in \operatorname{argmax}_{\theta \in \Theta} F(\theta, L)$ in any symmetric long-run equilibrium.

Economy M In Economy M, consumption good producers use the production function $\alpha^{-\alpha}(1-\alpha)^{\alpha-1}\left[F(\theta, L)\right]^{\alpha}\left[q(\theta)\right]^{1-\alpha}$,[44] where $q(\theta)$ denotes the quantity of an intermediate good supplied and produced at per unit cost of $(1-\alpha)$ (in terms of the consumption good) by a technology monopolist, which embodies the technology θ the monopolist chooses. This formulation follows the monopolistic competition approach in the endogenous growth literature, where monopolistic rents provide the incentive to conduct research.

For a fixed technology $\theta \in \Theta$ and a given aggregate supply $L \in \mathbb{R}^M_{++}$, a short-run equilibrium consists of (i) an input price vector p, (ii) a price for the monopolist's intermediate good, (iii) input quantities L^i, and (iv) a quantity of the intermediate good $q(\theta)$, such that all input markets and the intermediate good market clear, consumption good producers maximize profits taking θ as given, and the monopolist maximizes profits taking θ as given. Assuming symmetry, deriving consumption good producers' demand for the intermediate good, and maximizing the monopolist's profits shows that the monopolist supplies a quantity of $q(\theta) = F(\theta, L)(1-\alpha)/\alpha$. With inverse demand for non-technology inputs, this yields $p = D_L F(\theta, L)$ for any symmetric short-run equilibrium.

For a given aggregate supply $L \in \mathbb{R}^M_{++}$, a long-run equilibrium consists of (i) an input price vector p, (ii) a price for the monopolist's intermediate good, (iii) input quantities L^i, (iv) a quantity of the intermediate good $q(\theta^\star)$, and (v) a technology $\theta^\star \in \Theta$, such that all input markets and the intermediate good market clear, consumption good producers maximize profits taking θ as given, and the monopolist maximizes profits. As in a symmetric short-run equilibrium, prices are given by $p = D_L F(\theta^\star, L)$ in any symmetric long-run equilibrium, where $\theta^\star$ maximizes monopoly profits. Since monopoly profits at a given θ are $(1-\alpha)F(\theta, L)$, $\theta^\star \in \operatorname{argmax}_{\theta \in \Theta} F(\theta, L)$.

Economy O Economy O is as Economy M, only that consumption good producers produce according to $\alpha^{-\alpha}(1-\alpha)^{\alpha-1}\left[F(\theta, L)\right]^{\alpha}\sum_{k=1}^{K}\left[q_k(\theta_k)\right]^{1-\alpha}$, and there are K differ-

[44] The multiplication by $\alpha(1-\alpha)^{\alpha-1}$ is without loss in generality, because we can always adjust the definition of F accordingly.

ent technology firms, each of them being the monopoly supplier of an intermediate good q_k. Technology $\theta = (\theta_1, ..., \theta_K)$ is a composite of the K different technologies chosen by the technology firms, which are embodied in their intermediate goods.

The definitions of short-run and long-run equilibrium are analogous to Economy M, taking into account that each of the K technology firms maximizes profits individually. To derive conditions for equilibrium prices, again derive demand for intermediate goods and solve the profit maximization problem of technology firms. For a symmetric equilibrium, we obtain that $q_k(\theta_k) = F(\theta, L)(1 - \alpha)/\alpha$ for all k, analogous to the result in Economy M. Thus, at fixed θ, input prices must satisfy $p = D_L F(\theta, L)$ in any symmetric short-run equilibrium (by the same reasoning as for Economy M).

In long-run equilibrium, any technology firm chooses its θ_k to maximize profits, which again are proportional to $F(\theta, L)$ for all k. So, the long-run equilibrium technology must satisfy $\theta_k \in \text{argmax}_{\theta_k \in \Theta_k} F(\theta, L)$ for all k. If for any θ and L, there exists a short-run equilibrium, then there exists a long-run equilibrium with $\theta \in \text{argmax}_{\theta \in \Theta} F(\theta, L)$ for any L.

B. A Dynamic Model of Directed Technical Change

This section presents the generalized version of the standard dynamic model of directed technical change used in section 3.2 of the main text. The model uses the microfoundation from Acemoglu (1998), but does not assume a specific functional form for aggregate production. In particular, technologies are not restricted to be factor-augmenting.

The model consists of a household sector, a production sector and a research sector. Time is continuous and runs from 0 to infinity.

B.1. Household Sector

The household sector is modeled as a representative, infinitely-lived household. The household supplies the input factors $L \in \mathbb{R}^M_{++}$ inelastically at each time t. It chooses consumption C_t and savings $\dot{S}_t$ for each t to maximize lifetime utility

$$U_t \equiv \int_t^\infty e^{-\rho\tau} \frac{C_\tau^{1-\gamma} - 1}{1 - \gamma} d\tau$$

subject to the flow budget constraint (the price of the final good is normalized to one)

$$\dot{S}_t = \sum_{i=1}^{M} p_{it} L_i + \pi_t + r_t S_t - C_t.$$

The notation is as follows:ρ denotes the discount rate, $1/\gamma$ is the intertemporal elasticity of substitution, r_t is the interest rate, p_{it} denotes the return to input factor i, and π_t denotes total firm profits. Note that the entire net income of the economy accrues to the household,[45] hence the budget constraint is equivalent to the economy's aggregate resource constraint.

B.2. Production Sector

The production sector consists of final good producers and intermediate good producers.

The final good is produced by competitive firms according to the production function $F(\theta_t, L)$, where θ_t is an N-dimensional vector with (real) entries $\theta_{it} = \int_0^1 q_{i\kappa t}^{1-a} x_{i\kappa t}^{a}\, d\kappa$. The $x_{i\kappa t}$ are intermediate goods, $q_{i\kappa t}$ denotes the quality level at which the intermediate is available at time t.[46] This is a standard formulation of intermediate goods aggregation in Schumpeterian models of endogenous growth. Final good producers demand intermediates $x_{i\kappa t}$ and the exogenous inputs L to maximize profits. I impose the following assumption on the production function F.

Assumption 6. *The aggregate production function* $F : \mathbb{R}_{+}^{N} \times \mathbb{R}_{++}^{M} \to \mathbb{R}$, $(\theta, L) \mapsto F(\theta, L)$, *satisfies:*

1. F is C^2 and has strictly positive first-order partial derivatives on the interior of

[45] As usual in directed technical change models – and in many R&D-based endogenous growth models – some part of gross output is used as an input to the production of intermediate goods. Hence gross output exceeds the net income which eventually accrues to the household.

[46] As these model elements are well established, I use a shortcut here. More comprehensively, θ_{it} is a CES-aggregate of intermediate goods $y_{i\kappa t}$, $\theta_{it} = \int_0^1 y_{i\kappa t}^{a}\, d\kappa$, and each $y_{i\kappa t}$ can be produced by different firms at different quality levels:

$$y_{i\kappa t} = \sum_{j=0}^{n_{i\kappa t}} \left(\frac{q_{i\kappa t}}{\lambda^j} \right)^{\frac{1-a}{a}} x_{i\kappa t}^{(j)}.$$

Since all producers of intermediate good $i\kappa$ will have the same constant marginal costs, only the producer with the highest quality is active in equilibrium, such that $y_{i\kappa t} = q_{i\kappa t}^{(1-a)/a} x_{i\kappa t}^{(0)}$, which yields (dropping the superscript of $x_{i\kappa t}^{(0)}$):

$$\theta_{it} = \int_0^1 q_{i\kappa t}^{1-a} x_{i\kappa t}^{a}\, d\kappa.$$

its domain.

2. F is linear homogeneous in θ and homogeneous of degree $1-a \in (0,1)$ in L.

3. F is strictly quasiconcave in θ and $F\left(X_1^a, X_2^a, ..., X_N^a, L\right)$ is strictly quasiconcave in $(X_1, X_2, ..., X_N, L)$.

4. If $\left\{\theta^{(n)}\right\}_{n\in\mathbb{N}}$ is a sequence of technology inputs in $\mathbb{R}^N_{++}$ with $\theta_i^{(n)}/\theta_j^{(n)} \to 0$ for some i, j, then there exists a k with $\theta_k^{(n)}/\theta_j^{(n)} \to 0$ such that $D_{\theta_k}F(\theta^{(n)}, L) \to \infty$.

The first part of the assumption is standard. Linear homogeneity in θ is required for constant growth; homogeneity of degree $1-a$ in L is required for constant returns to scale in rival inputs.[47] The assumptions of quasiconcavity guarantee that there exist unique equilibrium values θ_{it} at every t for given quality levels $q_{i\kappa t}$.[48] The final point of the assumption ensures together with strict quasiconcavity in θ that the model has a unique balanced growth path.

Before continuing with the production of intermediate goods and to assist the understanding of the environment, I show that the set-up described so far embeds two well-known special cases. First, let $F(\theta_t, L) = \theta_{1t}L_1^{1-a}$. Then, the model becomes the standard textbook model of Schumpeterian endogenous growth; there is no directed technical change, as technology is one-dimensional. Second, let

$$F(\theta_t, L) = \left[\left(\theta_{1t}L_1^{1-a}\right)^{(\sigma-1)/\sigma} + \left(\theta_{2t}L_2^{1-a}\right)^{(\sigma-1)/\sigma}\right]^{\sigma/(\sigma-1)}.$$

In that case, the model is the standard directed technical change model developed by Acemoglu (1998) and Kiley (1999), where technologies are factor-augmenting. Note that in both examples F satisfies Assumption 6.

Each intermediate good $x_{i\kappa t}$ is produced by a monopolist, who owns an infinitely-lived patent for the production of good $i\kappa$ at quality level $q_{i\kappa t}$. Once a higher quality level is developed, the patent becomes worthless as only the producer with highest quality is active in equilibrium (see footnote 46). To produce one unit of intermediate $i\kappa$ the monopolist uses χ_i units of the final good.

[47] Linear Homogeneity in θ and homogeneity of degree $1-a$ in L imply that aggregate production is linear homogeneous in L and the intermediate goods $x_{i\kappa}$.

[48] In more detail, joint concavity of F in L and the makeshift inputs X_i guarantees that final good producers' profit maximization problem will be concave in intermediate inputs $x_{i\kappa}$ and L. Concavity of F in θ ensures that combining the optimal choices of final and intermediate good producers leads to unique equilibrium values θ_{it}. This will become clear in subsection B.4.

B.3. Research Sector

Research firms invent new quality levels for each intermediate good. When intermediate $i\kappa$ is currently produced at quality $q_{i\kappa t}$, an innovation in sector $i\kappa$ allows to raise the quality to $\lambda q_{i\kappa t}$ ($\lambda > 1$). The patent to produce $i\kappa$ at the new quality level is sold by the innovating research firm to intermediate goods firms. There is free entry on both sides of the market for patents: any firm can choose to invest into research and any firm can choose to buy a patent.

Innovations occur stochastically: any unit of research effort spent on an innovation in sector $i\kappa$ results in an innovation with the flow probability of $\phi(z_{i\kappa t})$, where $z_{i\kappa t}$ is total research effort devoted to sector $i\kappa$. The overall flow probability of an innovation in sector $i\kappa$ is therefore given by $z_{i\kappa t}\phi(z_{i\kappa t})$. Any unit of research effort in sector $i\kappa$ (targeted at the innovation of quality level $\lambda q_{i\kappa t}$) requires spending of $\eta_i q_{i\kappa t}$ units of final good. The function ϕ is assumed to be strictly decreasing with elasticity $\epsilon_\phi(z_{i\kappa t}) > -1$ such that the flow rate $z_{i\kappa t}\phi(z_{i\kappa t})$ is strictly increasing in $z_{i\kappa t}$. Moreover, $\phi(z) \to \infty$ as $z \to 0$ and $\phi(z) \to 0$ as $z \to \infty$. Thereby, ϕ guarantees that there will always be a positive amount of research effort devoted to sector $i\kappa$ in equilibrium, which simplifies the analysis of equilibrium dynamics.[49]

B.4. Equilibrium

The household takes prices as given and maximizes lifetime utility. This leads to a standard Euler equation,

$$\hat{C}_t = \frac{r_t - \rho}{\gamma}\,, \tag{B.1}$$

and the transversality condition[50]

$$\lim_{t\to\infty} e^{-\int_0^t r_\tau\, d\tau} S_t = 0\,,$$

stating that the present value of total asset holdings in the infinitely far future must be zero. (Since all assets in positive net supply take the form of patents, this condition can also be written as $\lim_{t\to\infty} e^{-\int_0^t r_\tau\, d\tau} \sum_{i=1}^N \int_0^1 V_{i\kappa t}(q_{i\kappa t})\, d\kappa = 0$.)

[49]This formulation of the research process closely follows Acemoglu (1998).

[50]More accurately, the transversality condition is

$$\lim_{t\to\infty} e^{-\int_0^t r_\tau\, d\tau} S_t \leq 0.$$

In combination with a No-Ponzi-Game condition, $\lim_{t\to\infty} e^{-\int_0^t r_\tau\, d\tau} S_t \geq 0$, this leads to the condition given above.

Final good firms take intermediate good prices as given and maximize profits. So, in equilibrium each input factor is paid its marginal product, $p_{it} = D_{L_i} F(\theta_t, L)$ for all $i = 1, ..., M$, and demand for intermediate goods is

$$x_{i\kappa t} = a^{1/(1-a)} P_{i\kappa t}^{1/(a-1)} \left(D_{\theta_i} F(\theta_t, L)\right)^{1/(1-a)} q_{i\kappa t}$$

where $P_{i\kappa t}$ denotes the price of intermediate $i\kappa$.

As demand for their products is isoelastic, intermediate good firms charge a constant markup over their marginal cost: $P_{i\kappa t} = \chi_i / a$. They produce

$$x_{i\kappa t} = \left(a^2/\chi_i\right)^{1/(1-a)} \left(D_{\theta_i} F(\theta_t, L)\right)^{1/(1-a)} q_{i\kappa t}$$

and earn profits of

$$\pi_{i\kappa t} = \chi_i^{-a/(1-a)} a^{(1-a)/(1-a)} (1-a) \left(D_{\theta_i} F(\theta_t, L)\right)^{1/(1-a)} q_{i\kappa t} . \tag{B.2}$$

It follows that

$$\theta_{it} = \chi_i^{-\frac{a}{1-a}} a^{\frac{2a}{1-a}} Q_{it} \left(D_{\theta_i} F(\theta_t, L)\right)^{\frac{a}{1-a}} , \tag{B.3}$$

where $Q_{it} \equiv \int_0^1 q_{i\kappa t} \, d\kappa$ is the average quality level of intermediates using technology i.

Lemma 3. *Under assumption 6 there exists a unique vector $\theta_t \in \mathbb{R}_{++}^N$ solving equation (B.3) for given quality levels $Q_t \in \mathbb{R}_{++}^N$.*

Proof. Any θ_t that solves equation (B.3) for given Q_{it} is a critical point of the function $\tilde{F}(\theta) \equiv F(\theta, L) - a^{-1} \sum_{i=1}^N \chi_i Q_{it}^{(a-1)/a} \theta_{it}^{1/a}$. Since $\tilde{F}$ is strictly concave and C^2 on $\mathbb{R}_{++}^N$ (as F is concave in θ and C^2 by assumption 6), it has at most one critical point on $\mathbb{R}_{++}^N$.

Existence follows from the observation that $\tilde{F}$ must attain a global maximum on $\mathbb{R}_{++}^N$. To see this, first note that $a^{-1} \sum_{i=1}^N \chi_i Q_{it}^{(a-1)/a} \theta_{it}^{1/a}$ is of order $1/a > 1$ in each θ_{it} while $F(\theta, L)$ is at most of order 1 in each θ_{it}. So, for every sequence $\left\{\theta^{(n)}\right\}_{n \in \mathbb{N}}$ with $\theta_i^{(n)} \to \infty$ as $n \to \infty$ for some i, $\tilde{F}(\theta^{(n)})$ must approach $-\infty$. It follows that F must attain a maximum on $\mathbb{R}_{+}^N$.

Secondly, suppose the maximum is at some $\bar{\theta}$ with $\bar{\theta}_i = 0$ for at least one i. Then, there must exist a sequence $\left\{\theta^{(n)}\right\}_{n \in \mathbb{N}}$ in $\mathbb{R}_{++}^N$ with $\theta_i^{(n)} = \bar{\theta}_i$ if $\bar{\theta}_i > 0$ and $\theta_i^{(n)} \to 0$ if $\bar{\theta}_i = 0$, such that $\limsup_{n \to \infty} D_{\theta_i} \tilde{F}(\theta^{(n)}) \leq 0$ if $\bar{\theta}_i = 0$, which requires $\lim_{n \to \infty} D_{\theta_i} F(\theta^{(n)}, L) = 0$ if $\bar{\theta}_i = 0$. But this contradicts concavity of F in θ: for any N, N' with $\theta_i^{(N')} < \theta_i^{(N)}$ if $\bar{\theta}_i = 0$, concavity requires $D_{\theta_i} F(\theta^{(N')}, L) \geq D_{\theta_i} F(\theta^{(N)}, L)$ for at least one i with $\bar{\theta}_i = 0$. So, F attains a maximum on $\mathbb{R}_{++}^N$. □

The lemma verifies that optimal choices of final and intermediate good firms lead to a unique static equilibrium in the production sector.

Research firms sell patents for newly invented quality levels at a price equal to the present value of the patent, since entry to the demand side of the market for patents is free. The present value of a patent for producing intermediate $i\kappa$ at a new quality level is

$$V_{i\kappa t} = \int_t^\infty e^{-\int_t^\tau [r_s + z_{i\kappa s}\phi(z_{i\kappa s})]\, ds} \pi_{i\kappa\tau}\, d\tau \ . \tag{B.4}$$

The corresponding Hamilton-Jacobi-Bellman equation is given by

$$\dot{V}_{i\kappa t} = r_t V_{i\kappa t} + z_{i\kappa t}\phi(z_{i\kappa t}) V_{i\kappa t} - \pi_{i\kappa t} \ . \tag{B.5}$$

Since entry to the supply side of the market for patents is free, the cost of one unit of research effort must equal the expected return in each sector $i\kappa$:

$$\eta_i q_{i\kappa t} = \phi(z_{i\kappa t}) V_{i\kappa t} \ . \tag{B.6}$$

It can be shown that $z_{i\kappa t} = z_{it}$ holds for all κ, i at every point in time in equilibrium.[51] Using this result and equations (B.2) and (B.6) in equation (B.5) leads to

$$z_{it}\phi(z_{it}) = \phi(z_{it})\alpha_i \left(D_{\theta_i} F(\theta_t, L)\right)^{\frac{1}{1-a}} - r_t - \epsilon_\phi(z_{it})\hat{z}_{it} \ . \tag{B.7}$$

where $\alpha_i = \chi_i^{-a/(1-a)} a^{(1-a)/(1-a)} (1-a)/\eta_i > 0$.

Together with the Euler equation (B.1), the aggregate resource constraint, and the fact that $\hat{Q}_{it} = (\lambda - 1) z_{it}\phi(z_{it})$[52], equation (B.7) governs the equilibrium dynamics of the model.

To close the derivation of equilibrium conditions, note that the aggregate resource constraint can be written as $F^{net}(\theta_t, L) = C_t + \sum_{i=1}^N \eta_i Q_{it} z_{it}$, where $F^{net}(\theta_t, L) \equiv F(\theta_t, L) - \sum_{i=1}^N \int_0^1 \chi_i x_{i\kappa t}\, d\kappa$ denotes aggregate output net of the goods spent on the production of intermediate goods. For the right-hand side of the resource constraint it was used that all (net) savings are invested into research: $\dot{S}_t = \sum_{i=1}^N \eta_i Q_{it} z_{it}$ (credit market clearing).

Finally, I reduce the set of equilibrium conditions using the results for intermediate

[51] Consider κ, κ' and suppose that $z_{i\kappa t} > z_{i\kappa' t}$. Since equation (B.7) holds for $z_{i\kappa t}$, this implies that $\hat{z}_{i\kappa t} > \hat{z}_{i\kappa' t}$ under the assumption that ϵ_ϕ is constant. This, in turn preserves the initial inequality $z_{i\kappa\tau} > z_{i\kappa'\tau}$ on some right neighborhood $[t, t+\delta)$ of t. But then, repeating the argument, it appears that $z_{i\kappa\tau} > z_{i\kappa'\tau}$ for all $\tau > t$. Via equation (B.6) this implies $V_{i\kappa\tau}/q_{i\kappa\tau} > V_{i\kappa'\tau}/q_{i\kappa'\tau}$ for all $\tau > t$, but via equation (B.4): $V_{i\kappa\tau}/q_{i\kappa\tau} < V_{i\kappa'\tau}/q_{i\kappa'\tau}$ for all $\tau > t$, a contradiction.

[52] The term $z_{it}\phi(z_{it})$ gives the flow probability of innovations for each intermediate $i\kappa$. The result for the (deterministic) growth rate of average quality Q_{it} relies on a law of large numbers. I omit its derivation as it is standard in Schumpeterian models of endogenous growth.

firm profits (B.2) and the definition of the value of a patent (B.4) to write the free-entry condition (B.6) as

$$1 = \phi(z_{it}) \int_t^{\infty} e^{-\int_t^{\tau} [r_s + z_{is}\phi(z_{is})]\, ds} \alpha_i \left(D_{\theta_i} F(\theta_\tau, L)\right)^{\frac{1}{1-a}} \, d\tau \ . \tag{B.8}$$

An equilibrium in this environment is given by time paths for technology inputs θ_t, for quality levels Q_t (the vector of Q_{it}s), for research effort z_t (the vector of z_{it}s), for consumption C_t, and for the interest rate r_t, which satisfy the derived Bellman type equation (B.7), the derived free-entry condition (B.8), the Euler equation (B.1), the equation for $\hat{Q}_{it}$, equation (B.3) linking Q_t and θ_t, the aggregate resource constraint and the household's transversality condition.

B.5. Balanced Growth Path

A balanced growth path of the model is defined as follows:

Definition 5. A *balanced growth path* is an equilibrium in which technology θ_t, aggregate output $F(\theta_t, L)$, and consumption C_t grow at the same, constant rate.

On a balanced growth path all technology inputs, aggregate output, and consumption grow at the same rate g_θ. It follows that quality levels Q_i must grow at rate g_θ as well and the interest rate r must be constant. Equations (B.1) and (B.7) become:

$$z\phi(z) = \phi(z)\alpha_i \left(D_{\theta_i} F(\theta_t^{bgp}, L)\right)^{\frac{1}{1-a}} - r \ \forall i \tag{B.9}$$

$$\gamma g_\theta = r - \rho \tag{B.10}$$

$$g_\theta = (\lambda - 1) z\phi(z), \tag{B.11}$$

with θ_t^{bgp} the path of technology.[53] Equation (B.9) provides a characterization of technology on a balanced growth path.

Compare this characterization to the first-order conditions associated with the Lagrangian of the maximization problem $\max_{\theta \in S_\alpha^N} F(\theta, L)$ with

$$S_\alpha^N = \left\{ \theta \in \mathbb{R}_+^N | \sum_{i=1}^{N} (1/\tilde{\alpha}_i)\theta_i = 1 \right\}$$

[53]The transversality condition is satisfied if $\rho > (1-\gamma)r$, which I assume henceforth.

and $\tilde{\alpha}_i = \alpha_i^{1-a}$:

$$\tilde{\alpha}_i D_{\theta_i} F(\theta, L) = \lambda \; \forall i \tag{B.12}$$

$$\sum_{i=1}^{N} (1/\tilde{\alpha}_i)\theta_i - 1 = 0, \tag{B.13}$$

where λ denotes the Lagrange-multiplier. Denote by $\theta^\star(L)$ the supremum of the set $\text{argmax}_{\theta \in S_\alpha} F(\theta, L)$ and observe the following.

First, Assumption 6 implies that F is concave in θ, so the first order conditions (B.12) and (B.13) are necessary and sufficient for a maximum of $F(\theta, L)$ on S_α^N. Moreover, since F is strictly quasiconcave in θ, there exists a unique maximizer (existence follows from compactness of S_α^N and continuity of F). Finally, the Inada type condition in point 4 of Assumption 6 ensures that the unique maximizer is in the interior of S_α^N.

Second, for any θ_t^{bgp} satisfying equation B.9, choose $\lambda = ((z\phi(z) + r)/\phi(z))^{1-a}$ and $s = \left(\sum_{i=1}^{N} (1/\tilde{\alpha}_i)\theta_{t,i}^{bgp}\right)^{-1}$ and note that $s\theta_t^{bgp}$ and λ satisfy the first order conditions (B.12) and (B.13). With the first observation, it follows that for any technology value on a balanced growth path, θ_t^{bgp}, there exists a value $s > 0$ such that $s\theta_t^{bgp} = \theta^\star(L)$.

Third, given $\theta^\star(L)$, use $\theta^\star(L)$ in equation (B.9) to determine z, r, and g_θ from equations (B.9) to (B.11) and construct the corresponding balanced growth path. It is easily verified that, given $\theta^\star(L) \in S_\alpha^N$, there exists a unique solution (z, r, g_θ) to equations (B.9) to (B.11). Existence of $\theta^\star(L)$, then, implies existence of a balanced growth path. With the second observation, uniqueness of $\theta^\star(L)$ implies uniqueness of the balanced growth path. This proves Lemma 2 in the main text.

C. Results from Monotone Comparative Statics

I briefly present the results from the theory of monotone comparative statics on which the global version of the LeChatelier Principle for Relative Demand is based.

The main version of the global principle, presented in Theorem 2, uses Theorem 4 from Milgrom and Shannon (1994), which says that the solution correspondence of a parametrized lattice programming problem is increasing in the choice set and in the parameter if and only if the objective function is quasisupermodular and satisfies the single crossing property in the choice variables and the parameter.

The maximizer sets are compared in the strong set order: for two subsets A and B of a lattice X, A dominates B in the *strong set order*, $A \geq_{SSO} B$, if for any $a \in A$ and $b \in B$, $a \vee b \in A$ and $a \wedge b \in B$. Quasisupermodularity and the single crossing property are defined in Section 2.2 of the main text.

Theorem (Milgrom and Shannon 1994)**.** *Let X be a lattice and S a partially ordered set. Consider a family of functions $\{f(\cdot;s)\}_{s\in S}$ with $f : X \times S \to \mathbb{R}$. Then,*

$$\operatorname*{argmax}_{x\in Y'\subset X} f(x;s') \leq_{SSO} \operatorname*{argmax}_{x\in Y''\subset X} f(x;s'') \quad \text{if } s' < s'' \text{ and } Y' \leq_{SSO} Y''$$

if and only if $f(x;s)$ is quasisupermodular in x and has the single crossing property in $(x;s)$.

Remark 3 says that the single crossing property in Theorem 2 can be relaxed. Indeed, when the choice set $Y \subset X$ is a sublattice of X and fixed, the single crossing property in Milgrom and Shannon's theorem can be replaced by the weaker concept of interval dominance order proposed by Quah and Strulovici (2009).

A family of functions $\{f(\cdot;s)\}_{s\in S}$ with $f : X \times S \to \mathbb{R}$, X a lattice, and S a partially ordered set obeys the *interval dominance order* if $f(x'';s') \geq (>) f(x';s') \Rightarrow f(x'';s'') \geq (>) f(x';s'')$ for all $s' < s''$ and $x' < x''$ which satisfy $f(x'';s') \geq f(x;s')$ for all $x \in [x',x''] = \{x \in X | x' \leq x \leq x''\}$.

Quah and Strulovici (2009) only consider problems where the choice set is a chain, so I prove the result required for Remark 3 here.

Lemma 4. *Let X be a lattice and S a partially ordered set. Suppose the function family $\{f(\cdot;s)\}_{s\in S}$ with $f : X \times S \to \mathbb{R}$ obeys the interval dominance order and $f(x;s)$ is quasisupermodular in x for all s. Then:*

$$\operatorname*{argmax}_{x\in X} f(x;s') \leq_{SSO} \operatorname*{argmax}_{x\in X} f(x;s'') \quad \text{if } s' < s''.$$

Proof. Take any $x' \in \operatorname{argmax}_{x\in X} f(x;s')$ and $x'' \in \operatorname{argmax}_{x\in X} f(x;s'')$.

We have $f(x' \wedge x'';s') \leq f(x';s')$ and hence, by quasisupermodularity, $f(x' \vee x'';s') \geq f(x'';s')$. Then, for any $x \in [x'', x' \vee x'']$, $x \vee x' = x' \vee x''$; to see this, note that $x, x' \leq x' \vee x'' \Rightarrow x \vee x' \leq x' \vee x''$ and $x', x'' \leq x \vee x' \Rightarrow x' \vee x'' \leq x \vee x'$. So, by the same argument as for x'': $f(x' \vee x'';s') \geq f(x;s')$ for all $x \in [x'', x' \vee x'']$. Finally, by the interval dominance order, $f(x' \vee x'';s'') \geq f(x'';s'')$ and hence $x' \vee x'' \in \operatorname{argmax}_{x\in X} f(x;s'')$.

Similarly, we have $f(x' \vee x'';s'') \leq f(x'';s'')$ and hence, by quasisupermodularity, $f(x' \wedge x'';s'') \geq f(x';s'')$. Now, suppose that $f(x' \wedge x'';s') < f(x';s')$. Since $f(x;s') \leq f(x';s')$ for any $x \in [x' \wedge x'', x']$, the interval dominance order implies $f(x' \wedge x'';s'') < f(x';s'')$, a contradiction. Thus, $f(x' \wedge x'';s') \geq f(x';s')$ and $x' \wedge x'' \in \operatorname{argmax}_{x\in X} f(x;s')$. □