Appendix to ‘Endogenous Growth, Green Innovation and GDP Deceleration in a World with Polluting Production Inputs’

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Contents

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A Appendix to section 3: The laissez-faire equilibrium

A.1 The representative household

The representative household earns income from labor and asset holding which he spends on consumption and the acquisition of new assets. The budget constraint is

$$c_t L + \dot{A}_t = r_t A_t + \int_0^1 w_{Xit} L_{Xit} di + w_{Yt} L_{Yt} + w_{Dt} L_{Dt}, \tag{A.1}$$

where A_t denotes asset holdings and w_{Xit}, w_{Yt} and w_{Dt} the wage rates for labor in intermediate production, production of the consumption good and research. The interest rate is denoted by r_t.

For notational convenience, we rewrite the utility function (1) as function of the pollution stock S rather than environmental quality E, using (2):

$$U = \int_0^\infty e^{-\rho t} \left(\frac{\sigma_c}{\sigma_c - 1} c_t^{\frac{\sigma_c - 1}{\sigma_c}} - \psi \frac{\sigma_E}{1 - \sigma_E} S_t^{\frac{1-\sigma_E}{\sigma_E}} \right) L dt \tag{A.2}$$

The household maximizes (A.2) by choosing the paths for consumption, labor and asset holding while taking pollution accumulation as given. He takes into account the budget-constraint (A.1) and must satisfy the no-Ponzi condition

$$\lim_{t \to \infty} \left(e^{-\int_0^t r_v dv} A_t \right) \geq 0.$$

The current-value Hamiltonian function is:

$$\begin{aligned} H \;=\; & \left(\frac{\sigma_c}{\sigma_c - 1} c_t^{\frac{\sigma_c - 1}{\sigma_c}} - \psi \frac{\sigma_E}{1 - \sigma_E} S_t^{\frac{1-\sigma_E}{\sigma_E}} \right) L \\ & + v_{At} \left(r_t A_t + \int_0^1 w_{Xit} L_{Xit} di + w_{Yt} L_{Yt} + w_{Dt} L_{Dt} - c_t L \right) \\ & + \lambda_{Lt} \left(L - \left(\int_0^1 L_{Xit} di + L_{Yt} + L_{Dt} \right) \right) \end{aligned}$$

v_{At} is the current-value costate variable of assets A in t and λ_{Lt} the Lagrange-multiplier of the constraint on labor. The first-order conditions according to Pontryagin's maximum principle are:

$$\frac{\partial H}{\partial c_t} = 0 \Leftrightarrow v_{At} = c_t^{\frac{-1}{\sigma_c}} \tag{A.3}$$

$$\frac{\partial H}{\partial L_{Xit}} = 0 \Leftrightarrow v_{At} w_{Xit} = \lambda_{Lt}$$

$$\frac{\partial H}{\partial L_{Yt}} = 0 \Leftrightarrow v_{At} w_{Yt} = \lambda_{Lt}$$

$$\frac{\partial H}{\partial L_{Dt}} = 0 \Leftrightarrow v_{At} w_{Dt} = \lambda_{Lt}$$

$$\frac{\partial H}{\partial A_t} = \rho v_{At} - \dot{v}_{At} \Leftrightarrow v_{At} r_t = \rho v_{At} - \dot{v}_{At} \tag{A.4}$$

$$\frac{\partial H}{\partial v_{At}} = \dot{A}_t \Leftrightarrow \dot{A}_t = r_t A_t + \int_0^1 w_{Xit} L_{Xit} di + w_{Yt} L_{Yt} + w_{Dt} L_{Dt} - c_t L \tag{A.5}$$

$$\frac{\partial H}{\partial \lambda_{Lt}} = 0 \Leftrightarrow L = \int_0^1 L_{Xit} di + L_{Yt} + L_{Dt}$$

The first-order conditions for the different types of labor can only be satisfied simultaneously, if firms in the different sectors of intermediate production as well as firms in final good production and research all offer the same wage. The household is then indifferent about the allocation of his labor supply.

The first-order condition for assets, A_t, can be restated as $\widehat{v}_{At} = \rho - r_t$. Log-differentiating both sides of the first-order condition for consumption yields

$$\frac{-1}{\sigma_c}\widehat{c}_t = \widehat{v}_{At}.$$

By substituting the expression for $\widehat{v}_{At}$, we obtain the standard Euler-equation for per capita consumption:

$$\widehat{c}_t = \sigma_c \cdot (r_t - \rho) \tag{A.6}$$

A.2 Production

The production function for the consumption good is given by (5). Firms maximize profits over L_Y and X_i, taking the wage rate w_{Yt} and the prices p_{it} of the intermediates in sectors $i \in [0,1]$ as given. We normalize the price of the consumption good to one. The first order condition for L_Y yields the implicit labor demand function

$$w_{Yt} = (1-\alpha)\, L_{Yt}^{-\alpha} \int_0^1 X_{it}^{\alpha} Q_{it}^{1-\alpha} di. \tag{A.7}$$

From the first-order condition for X_i, the following demand function for intermediate i is derived:

$$X_{it}^{d}(p_{it}) = \left(\frac{\alpha}{p_{it}}\right)^{\frac{1}{1-\alpha}} Q_{it} L_{Yt} \tag{A.8}$$

Each unit of the intermediate is produced with the production function (7):

$$X_{it} = \varphi L_{Xit} Q_t$$

At equilibrium, the wage in intermediate production must be the same in every sector i, so that marginal costs $MC_t = (1/\varphi)\cdot(w_{Xt}/Q_t)$ are the same for goods with different productivity levels. On the other hand, final good producers' demand is larger for more productive intermediates. It follows that only the owner of the patent for the intermediate design with the highest productivity will be producing in sector i, as he can always choose a price so that the firm with the next highest productivity level cannot break even. For the rest of this subsection, the firm index j is therefore omitted.

The intermediate good in sector i is sold at a price p_{it} to firms in the final good sector. The monopoly producer chooses p_{it} to maximize profits

$$\pi_{it}^{X}(p_{it}) = (p_{it} - MC_t)X_{it},$$

taking into account the demand function (A.8). The profit maximizing monopoly price is given by a constant mark-up over marginal costs for all i[1]:

$$p_t = \frac{1}{\alpha\varphi} \cdot (w_{Xt}/Q_t)$$

[1] Monopoly pricing prevails under certain restrictions on model parameters which we derive in section A.3.

The wage rate at equilibrium is obtained by substituting (A.8) in (A.7), using the fact that wages from intermediate and final good production as well as research must be equal in equilibrium if all three types of labor are to be provided by the household. The equilibrium wage is:

$$w_{Xt} = w_{Yt} = w_{Dt} = (1-\alpha)^{1-\alpha} \alpha^{2\alpha} (\varphi)^{\alpha} Q_t \tag{A.9}$$

We then derive the quantity of intermediates produced in sector i as function of the amount of labor employed in final good production, for any given sectoral level of productivity, from (A.8):

$$X_{it} = \frac{\alpha^2}{1-\alpha} \varphi L_{Yt} Q_{it} \tag{A.10}$$

Monopoly profits in sector i in period t are:

$$\pi_{it}^X = \frac{(1-\alpha)^{1-\alpha}}{\alpha} \alpha^{2(1+\alpha)} \varphi^{\alpha} L_{Yt} \cdot Q_{it} \tag{A.11}$$

The aggregate quantity X_t of intermediates is

$$X_t = \int_0^1 X_{it} di = \frac{\alpha^2}{(1-\alpha)} \varphi L_{Yt} Q_t, \tag{A.12}$$

where we used the definition $\int_0^1 Q_{it} di := Q_t$ of aggregate productivity.

A.3 Research

At time t, researcher j in sector i chooses l_{Dijt}, q_{ijt} and b_{ijt} to maximize expected profits from R&D. These consist of the profit flow he expects to receive as a monopolist in intermediate production less of research labor costs.

In every period and every sector, the exogenous arrival rate of innovations for the individual researcher is μ. If researcher j succeeds in innovating, he changes the productivity level in sector i from Q_{it} to $(q_{ijt}+1) \cdot Q_{it}$. After the innovation, the productivity level remains constant until the next innovation occurs and the monopoly producer is replaced by the new innovator.

The probability per unit of time of being replaced as the monopolist in sector i is exogenously given from the perspective of researcher j in every period $v > t$ and increases in the mass n_{iv} of research units active in sector i at time v. More precisely, innovations in every sector i follow a Poisson-process with arrival-rate $\mu_{iv} = \mu \cdot n_{iv}$. The probability that the incumbent monopolist is still producing in period $s > t$ is then given by $P(s) = e^{-\int_t^s \mu_{iv} dv}$. His profits in period s can be deduced from (A.11), substituting the after-innovation productivity level $(q_{ijt}+1) \cdot Q_{it}$ for Q_{it}.

Expected discounted lifetime-profits are:

$$\begin{aligned} E\left[V_{ijt}(q_{ijt})\right] &= \textstyle\int_t^{\infty} \pi_{ijs}^X (q_{ijt}) \cdot P(s) e^{-\int_t^s r_v dv} ds \\ &= \frac{(1-\alpha)^{1-\alpha}}{\alpha} \alpha^{2(1+\alpha)} \varphi^{\alpha} (q_{ijt}+1) \cdot Q_{it} \textstyle\int_t^{\infty} L_{Ys} e^{-\int_t^s (r_v + \mu_{iv}) dv} ds \end{aligned} \tag{A.13}$$

Expected research profits are obtained by substracting research costs $w_{Dt} l_{Dijt}$:

$$E\left[\pi_{ijt}^D(q_{ijt}, b_{ijt})\right] = \mu E\left[V_{ijt}(q_{ijt})\right] - w_{Dt} l_{Dijt}(q_{ijt}, b_{ijt}) \tag{A.14}$$

Labor l_{Dijt} is given by (9) and the wage w_{Dt} by (A.9).

Researcher j maximizes (A.14) by choosing q_{ijt} and b_{ijt}. Reducing the pollution intensity of intermediates by increasing B_i is costly but does not increase profits $E[V_{ijt}]$. Therefore $b_{ijt} = 0$ for all i, j, t so that the pollution intensity of intermediates is constant under laissez-faire. The first-order condition for q_{ij} can, after simplification, be written as:

$$\alpha\mu\int_t^\infty L_{Ys}e^{-\int_t^s (r_v+\mu_{iv})dv}ds - 2q_{ijt} = 0 \tag{A.15}$$

The equation still depends on n_i through the sectoral arrival rate μ_i. To determine q_{it} and n_{it}, it must be taken into account that expected research profits in every sector i have to be zero at equilibrium. The zero profit condition is:

$$\int_t^\infty L_{Ys}e^{-\int_t^s (r_v+\mu_{iv})dv}ds = \frac{q_{ijt}^2 + d}{(1+q_{ijt})\alpha\mu} \tag{A.16}$$

From (A.15) with (A.16), we determine the equilibrium value

$$q_{ijt}^{\mathrm{LF}} = q^{\mathrm{LF}} = \sqrt{1+d} - 1 \tag{A.17}$$

of q_{ijt}. q^{LF} is constant over time and across sectors. It increases in the entry cost parameter d because less entry lowers the probability of being replaced by the next innovator and therefore increases marginal profits from productivity improvements[2].

Because q^{LF} is constant, the integral $\int_t^\infty L_{Ys}e^{-\int_t^s (r_v+\mu_{iv})dv}ds$ on the left-hand side of the free-entry condition (A.16) must be independent of t. Setting the time derivative of the integral to zero shows that the integral must be equal to $\frac{L_{Yt}}{r_t+\mu_{it}}$. We show in the next section, that L_{Yt}, r_t and n_{it} and thus the ratio $\frac{L_{Yt}}{r_t+\mu_{it}}$ are constant even if the economy is not on a balanced growth path. After substituting $\frac{L_{Yt}}{r_t+\mu_{it}}$ for the integral in equation (A.16) and the equilibrium value of q, (A.17), on the right-hand side, we can solve the free-entry condition for n_{it}:

$$n_{it} = n_t = \frac{1}{2}\frac{1}{\sqrt{1+d}-1}\cdot\alpha L_{Yt} - \frac{r_t}{\mu}. \tag{A.18}$$

A.4 General equilibrium

A.4.1 The market value of firms

Every unit of assets A in our model corresponds to a share of the market value of firms in the intermediate sector. The total stock of the representative household's assets at the beginning of period t must therefore equal the aggregate market value of firms before innovation. In each sector i, only the firm with the highest productivity level Q_{it} is active in production. The before-innovation market value of this firm can be derived from (A.13), substituting Q_{it} for the after-innovation productivity level $(q_{ijt}+1)\cdot Q_{it}$. To obtain the aggregate market value V_t of firms, we take the integral over all sectors and use (A.16) with (A.17) to replace $\int_t^\infty L_{Ys}e^{-\int_t^s (r_v+\mu_{iv})dv}ds$:

$$\begin{aligned} V_t &= \int_0^1 E[V_{ijt}]\,di \\ &= 2\frac{(1-\alpha)^{1-\alpha}\alpha^{2\alpha}\varphi^\alpha}{\mu}\left(\sqrt{1+d}-1\right)Q_t \end{aligned} \tag{A.19}$$

The market value is proportional to the economy-wide productivity level Q_t.

[2] In the analysis, it has been assumed that the monopoly price is smaller than the limit price. This will be the case whenever $p^{\mathrm{mon}} < \left(q^{\mathrm{LF}}+1\right)\cdot(1/\varphi)\,(w_{Xt}/Q_t)$ which is equivalent to $d > \frac{1}{\alpha^2} - 1$.

A.4.2 Labor market clearing

We use (A.18) along with the labor market constraint (4) and equation (A.12) to find the allocation of labor between final good production, intermediate production and research (L_{Yt}, L_{Xt}, L_{Dt}) and determine the mass n_t of research units in sector i for any given interest rate r_t. The equilibrium n_t is:

$$n_t = \frac{\frac{1}{2}L - \left(\frac{1}{\alpha} + \frac{\alpha}{1-\alpha}\right)\left(\sqrt{1+d}-1\right)\frac{r_t}{\mu}}{\left(\frac{1-\alpha}{\alpha} + \frac{\alpha}{1-\alpha}\right)\left(\sqrt{1+d}-1\right)+d} \tag{A.20}$$

The mass of research units is the same in every sector. It increases in the arrival rate μ for innovations and decreases in the interest rate r_t and the fixed labor requirement d.

A.4.3 Equilibrium growth

Taking into account that n_t and q^{LF} are the same for all research sectors and using the definition of the aggregate productivity index Q, the equation of motion (12) for Q simplifies to

$$\dot{Q}_t = \mu n_t q^{\text{LF}} Q_t.$$

Substituting (A.17) for q^{LF} and (A.20) for n_t, we obtain the productivity growth rate in period t as a function of the interest rate r_t:

$$\widehat{Q}_t = \mu \frac{\frac{1}{2}L - \left(\frac{1}{\alpha} + \frac{\alpha}{1-\alpha}\right)\left(\sqrt{1+d}-1\right)\frac{r_t}{\mu}}{\left(\frac{1-\alpha}{\alpha} + \frac{\alpha}{1-\alpha}\right)\left(\sqrt{1+d}-1\right)+d}\left(\sqrt{1+d}-1\right) \tag{A.21}$$

It follows from (A.12) that X_t and Q_t grow at the same rate at equilibrium because labor must be constant. From the resource constraint, it is obvious that c_t then also grows at the rate $\widehat{Q}_t$. We set (A.21) equal to (A.6) and solve for the equilibrium interest rate.

$$r^{\text{LF}} = \frac{\frac{1}{2}\frac{1}{\sigma_c}\mu L\left(\sqrt{1+d}-1\right) + \left(\left(\frac{1-\alpha}{\alpha} + \frac{\alpha}{1-\alpha}\right)\left(\sqrt{1+d}-1\right)+d\right)\rho}{\left(\frac{1-\alpha}{\alpha} + \frac{\alpha}{1-\alpha}\right)\left(\sqrt{1+d}-1\right)+d+\frac{1}{\sigma_c}\left(\frac{1}{\alpha} + \frac{\alpha}{1-\alpha}\right)\left(\sqrt{1+d}-1\right)^2} \tag{A.22}$$

With r^{LF}, equation (A.21) yields the equilibrium productivity growth rate

$$\widehat{Q}^{\text{LF}} = \frac{\frac{1}{2}\mu L - \left(\frac{1}{\alpha} + \frac{\alpha}{1-\alpha}\right)\left(\sqrt{1+d}-1\right)\rho}{\left(\frac{1-\alpha}{\alpha} + \frac{\alpha}{1-\alpha}\right)\left(\sqrt{1+d}-1\right)+d+\frac{1}{\sigma_c}\left(\frac{1}{\alpha} + \frac{\alpha}{1-\alpha}\right)\left(\sqrt{1+d}-1\right)^2}\left(\sqrt{1+d}-1\right). \tag{A.23}$$

$\widehat{Q}^{\text{LF}}$ is positive if and only if the representative household is sufficiently patient, i.e., if and only if $\rho < \overline{\rho}^{\text{LF}} := \frac{1}{2}\frac{\mu L}{\left(\frac{1}{\alpha}+\frac{\alpha}{1-\alpha}\right)\left(\sqrt{1+d}-1\right)}$.

The growth rates of c, X, Q and B are constant for any set of initial values for the state variables. Therefore growth in c, X, Q and B is balanced without transitional dynamics.

It follows from (8) that the pollution stock must increase at the same rate $\widehat{Q}^{\text{LF}}$ as intermediate quantity, productivity and consumption in the long run:

$$\widehat{S}^{\text{LF}}_{\infty} = \widehat{Q}^{\text{LF}}$$

However, contrary to the growth rates of the other variables, the growth rate of the pollution stock does not adjust to its balanced-growth level instantly if the relation between the state variables is not reconcilable with constant growth of the pollution stock initially.

A.5 Proof of proposition 1

1. **Existence and Uniqueness:** The path defined by initial states Q_0, B_0 and S_0, the initial values X_0 from (7) and $c_0 = Y_0$ from (6) as well as the growth rates $\widehat{c}^{\mathrm{LF}} = \widehat{Y}^{\mathrm{LF}} = \widehat{X}^{\mathrm{LF}} = \widehat{Q}^{\mathrm{LF}}$, $\widehat{B}^{\mathrm{LF}} = 0$ in every period t and $\widehat{S}_t^{\mathrm{LF}}$ from the pollution accumulation function (8) satisfies all the necessary conditions for an equilibrium as defined in section 3 of the paper. If the initial values Q_0, B_0 and S_0 for the state variables are such that with X_0 from (7), the pollution accumulation function (8) yields the balanced growth rate $\widehat{S}_\infty^{\mathrm{LF}}$ in $t = 0$, the path is characterized by balanced growth. The Hamiltonian function for the intertemporal maximization problem of the representative household is strictly concave in consumption and linear in all other variables. It follows that the household's maximization problem has a unique solution. The same is true for the static maximization problems in the R&D-sector as well as the production sectors for the consumption good and intermediates, which are concave as well. The path described in the text is therefore the unique laissez-faire equilibrium for $\underline{\rho}^{\mathrm{LF}} < \rho < \overline{\rho}^{\mathrm{LF}}$ and, if the initial values Q_0, B_0 and S_0 for the state variables are reconcilable with balanced growth, the unique balanced-growth equilibrium. For $\underline{\rho}^{\mathrm{LF}} < \overline{\rho}^{\mathrm{LF}} \leq \rho$, it still needs to be shown that $n_t = n_{it} = 0$ for all t is an equilibrium. $n = 0$ implies $\widehat{Q}^{\mathrm{LF}} = \widehat{c}^{\mathrm{LF}} = 0$. Setting $\widehat{c}^{\mathrm{LF}}$ in the Euler-equation and solving for r yields $r = \rho$. For $r = \rho$, equation (A.7) and the maximization problem in the R&D-sector yield $w_{Yt} > w_{Dt}$, which proves that $n = 0$ is an equilibrium for this value of r.

2. **Derivation of $\underline{\rho}^{\mathrm{LF}}$:** The critical value $\underline{\rho}^{\mathrm{LF}} := \frac{1}{2}\alpha(1-\alpha)\left(1 - \frac{1}{\sigma_c}\right)\mu L\left(1+d\right)^{-1/2}$ is derived from the transversality condition $\lim_{t\to\infty}\left(e^{-\rho t}v_{At}A_t\right) = 0$. Using (A.4), substituting $A_t = A_0 e^{\widehat{Q}^{\mathrm{LF}}\cdot t}$ with $A_0 = 2\frac{(1-\alpha)^{1-\alpha}\alpha^{2\alpha}\varphi^\alpha}{\mu}\left(\sqrt{1+d}-1\right)Q_0$ from (A.19) and taking into account that $r_t = r^{\mathrm{LF}}$ for all t shows that the condition can be simplified to $v_{A0}A_0 \lim_{t\to\infty} e^{-(r^{\mathrm{LF}} - \widehat{Q}^{\mathrm{LF}})t} = 0$. The transversality condition is satisfied if and only if $r^{\mathrm{LF}} - \widehat{Q}^{\mathrm{LF}} > 0$. With (A.22) and (A.23), the critical value $\underline{\rho}^{\mathrm{LF}}$ follows.

3. **Welfare comparison:** To prove that for convex disutility of pollution, a path without long-run growth would be welfare-improving, consider the utility function (A.2). For convex disutility of pollution ($\sigma_E < 1/2$), $\frac{1-\sigma_E}{\sigma_E}$ is at least one while $\frac{\sigma_c - 1}{\sigma_c}$ is smaller than one. Along the balanced-growth path, $\widehat{S}^{\mathrm{LF}} = \widehat{S}_\infty^{\mathrm{LF}} = \widehat{c}^{\mathrm{LF}}$. Instantaneous utility $u_t = \frac{\sigma_c}{\sigma_c - 1}c_t^{\frac{\sigma_c-1}{\sigma_c}} - \psi\frac{\sigma_E}{1-\sigma_E}S_t^{\frac{1-\sigma_E}{\sigma_E}}$ converges to $-\phi^S(S_t) = -\psi\frac{\sigma_E}{1-\sigma_E}S_t^{\frac{1-\sigma_E}{\sigma_E}}$ and declines persistently towards $(-\infty)$. The long-run growth rate is $\frac{1-\sigma_E}{\sigma_E}\widehat{S}_\infty^{\mathrm{LF}}$. Now assume instead that economic growth is given up in a period s: Consumption growth drops to zero instantly while pollution growth converges to zero over time. Initially, there is a loss in per-period-utility compared to the laissez-faire equilibrium. This loss is only transitory: In the long-run, the pollution stock is constant and so is utility, while utility decreases in the laissez-faire equilibrium. Therefore, from a certain time onwards, not growing yields a utility-gain in each period and the gain increases as $t \to \infty$. Because of the concavity of the utility from consumption and convexity of the disutility from pollution, the transitional welfare-loss is smaller, the later in time the regime-switch occurs and converges to zero as $s \to \infty$. Giving up economic growth in the long-run therefore yields an increase in intertemporal welfare.

B Appendix to section 4: The Social Planner's solution

B.1 Maximization problem

To see that the optimal q_{it} and b_{it} are the same for all sectors i, i.e. $q_{it} = q_t$ and $b_{it} = b_t$, note that the social planner chooses the step-size in every sector i so as to reach a given rate of change $\dot{Q}_t$ and $\dot{B}_t$ in the respective aggregate technology level with a minimum labor investment. From the equations of motion (12) and (13) for Q and B together with the R&D-cost function (9) we can conclude that the marginal gain of an increase in b_i and q_i, in terms of faster technological progress, and the additional amount of labor required increase in the sectorial technology levels Q_{it} and B_{it} in the same way. Therefore sectorial differences are irrelevant for the optimal choice of q_i and b_i.

The dynamic optimization problem then depends on aggregate variables only: From (9), with $\int_0^1 Q_{it}di = Q_t$, $\int_0^1 B_{it}di = B_t$ and $n_{it} = n_t$, the amount of labor allocated to research in period t is $L_{Dt} = n_t(q_t^2 + b_t^2 + d)$. To produce X_t units of intermediates requires $L_{Xt} = \frac{1}{\varphi}\frac{X_t}{Q_t}$ units of labor. The labor market constraint can be written as

$$L = \frac{1}{\varphi}\frac{X_t}{Q_t} + L_{Yt} + n_t(q_t^2 + b_t^2 + d). \tag{B.1}$$

The equations of motion (12) for Q and (13) for B are:

$$\dot{Q}_t = \mu n q_t Q_t \tag{B.2}$$

$$\dot{B}_t = \mu n b_t B_t \tag{B.3}$$

Given aggregate intermediate production X_t the decision over X_{it} is static. The planner optimally allocates a higher share of aggregate intermediate production to the sectors with higher productivity level so as to maximize Y_t. The optimal X_{it} is:

$$X_{it} = X_t\frac{Q_{it}}{Q_t} \tag{B.4}$$

With (B.4), the aggregate resource constraint can be rewritten as:

$$L_{Yt}^{1-\alpha}X_t^{\alpha}Q_t^{1-\alpha} = c_t L \tag{B.5}$$

The dynamic maximization problem is solved by finding the optimal paths for Q, B, S, c, X, L_Y, n, q and b subject to (8), (B.1), (B.2), (B.3) and the resource constraint (B.5). The current-value Hamiltonian is given by:

$$\begin{aligned}
H \;=\; & \left(\frac{\sigma_c}{\sigma_c - 1}c_t^{\frac{\sigma_c - 1}{\sigma_c}} - \psi\frac{\sigma_E}{1-\sigma_E}S_t^{\frac{1-\sigma_E}{\sigma_E}}\right)L \\
& + v_{St}\left(\frac{X_t}{B_t} - \delta S_t\right) \\
& + v_{Qt}\mu n_t q_t Q_t \\
& + v_{Bt}\mu n_t b_t B_t \\
& + \lambda_{Yt}\left(X_t^{\alpha}Q_t^{1-\alpha}L_{Yt}^{1-\alpha} - c_t L\right) \\
& + \lambda_{Lt}(L - \frac{1}{\varphi}\frac{X_t}{Q_t} - L_{Yt} - n_t(q_t^2 + b_t^2 + d))
\end{aligned}$$

where v_{St}, v_{Qt} and v_{Bt} are the shadow-prices of S_t, Q_t and B_t respectively and λ_{Yt} and λ_{Lt} are Lagrange-multipliers.

B.2 First-order conditions

The first-order conditions are:

$$\frac{\partial H}{\partial c_t} = 0 \Leftrightarrow \lambda_{Yt} = c_t^{-1/\sigma_c} \tag{B.6}$$

$$\frac{\partial H}{\partial X_t} = 0 \Leftrightarrow \frac{v_{St}}{B_t} + \lambda_{Yt}\alpha X_t^{\alpha-1} L_{Yt}^{1-\alpha} Q_t^{1-\alpha} - \lambda_{Lt}\frac{1}{\varphi Q_t} = 0 \tag{B.7}$$

$$\frac{\partial H}{\partial q_t} = 0 \Leftrightarrow v_{Qt}\mu n_t Q_t = 2\lambda_{Lt} n_t q_t \tag{B.8}$$

$$\frac{\partial H}{\partial b_t} = 0 \Leftrightarrow v_{Bt}\mu n_t B_t = 2\lambda_{Lt} n_t b_t \tag{B.9}$$

$$\frac{\partial H}{\partial n_t} = 0 \Leftrightarrow v_{Qt}\mu q_t Q_t + v_{Bt}\mu b_t B_t = \lambda_{Lt}\left(q_t^2 + b_t^2 + d\right) \tag{B.10}$$

$$\frac{\partial H}{\partial L_{Yt}} = 0 \Leftrightarrow \lambda_{Yt}(1-\alpha) X_t^{\alpha} Q_t^{1-\alpha} L_{Yt}^{-\alpha} = \lambda_{Lt} \tag{B.11}$$

$$\frac{\partial H}{\partial S_t} = \rho v_{St} - \dot{v}_{St} \Leftrightarrow -\psi S_t^{(1-2\sigma_E)/\sigma_E} L - \delta v_{St} = \rho v_{St} - \dot{v}_{St} \tag{B.12}$$

$$\frac{\partial H}{\partial Q_t} = \rho v_{Qt} - \dot{v}_{Qt}$$

$$\Leftrightarrow \quad v_{Qt}\mu n_t q_t + \lambda_{Yt}(1-\alpha) X_t^{\alpha} Q_t^{-\alpha} L_{Yt}^{1-\alpha} + \lambda_{Lt}\frac{X_t}{\varphi}\frac{1}{Q_t^2} = \rho v_{Qt} - \dot{v}_{Qt} \tag{B.13}$$

$$\frac{\partial H}{\partial B_t} = \rho v_{Bt} - \dot{v}_{Bt} \Leftrightarrow -v_{St}\frac{X_t}{B_t^2} + v_{Bt}\mu n_t b_t = \rho v_{Bt} - \dot{v}_{Bt} \tag{B.14}$$

$$\frac{\partial H}{\partial v_{St}} = \dot{S}_t \Leftrightarrow \frac{X_t}{B_t} - \delta S_t = \dot{S}_t \tag{B.15}$$

$$\frac{\partial H}{\partial v_{Qt}} = \dot{Q}_t \Leftrightarrow \mu n_t q_t Q_t = \dot{Q}_t \tag{B.16}$$

$$\frac{\partial H}{\partial v_{Bt}} = \dot{B}_t \Leftrightarrow \mu n_t b_t B_t = \dot{B}_t \tag{B.17}$$

$$\frac{\partial H}{\partial \lambda_{Yt}} = 0 \Leftrightarrow X_t^{\alpha} Q_t^{1-\alpha} L_{Yt}^{1-\alpha} = c_t L \tag{B.18}$$

$$\frac{\partial H}{\partial \lambda_{Lt}} = 0 \Leftrightarrow L = \frac{1}{\varphi}\frac{X_t}{Q_t} + L_{Yt} + n_t(q_t^2 + b_t^2 + d) \tag{B.19}$$

Further, the transversality conditions

$$\begin{aligned} \lim_{t\to\infty}\left(e^{-\rho t} v_{Qt} Q_t\right) &= 0 \\ \lim_{t\to\infty}\left(e^{-\rho t} v_{Bt} B_t\right) &= 0 \\ \lim_{t\to\infty}\left(e^{-\rho t} v_{St} S_t\right) &= 0 \end{aligned} \tag{TVC}$$

and the non-negativity constraints

$$Q_t, B_t, S_t, c_t, X_t, L_{Yt}, n_t \geq 0,\ \forall t$$

must be satisfied.

From the first-order conditions, four key equations crucial for the determination of the long-run optimum are derived: The condition (16) for *asymptotically-balanced growth* in the text follows from the first-order conditions for X and S: The first-order condition (B.7) for X yields a relation $\widehat{v}_{S\infty} = (1 - 1/\sigma_c)\,\widehat{c}_\infty + \widehat{B}_\infty - \widehat{X}_\infty$ between the growth rates of the marginal utility c_t^{-1/σ_c} of consumption and the shadow price v_S of pollution for $t \to \infty$. From the first-order condition (B.12) for the pollution stock, it follows that along an ABG path, the ratio $S_t^{(1-2\sigma_E)/\sigma_E}/v_{St}$ must be constant for v_S to grow at a constant rate. In the long run, v_S must therefore grow at the same rate as the (instantaneous) marginal disutility $\psi S^{(1-2\sigma_E)/\sigma_E}$ of pollution, i.e. $\widehat{v}_{S\infty} = ((1 - 2\sigma_E)/\sigma_E)\,\widehat{S}_\infty$. Setting this expression for $\widehat{v}_{S\infty}$ equal with the one obtained from (B.7) and rearranging, the following relation between long-run pollution growth and consumption growth is derived:

$$\frac{\sigma_c - 1}{\sigma_c}\widehat{c}_\infty = \frac{1 - \sigma_E}{\sigma_E}\widehat{S}_\infty + \left(\widehat{X}_\infty - \widehat{B}_\infty - \widehat{S}_\infty\right) \tag{16G}$$

Note that (16G) is a more general form of (16) in the proof of lemma 2 in the paper. In equation (16), it has been taken into account that $\widehat{S}_\infty = \widehat{X}_\infty - \widehat{B}_\infty$ if condition (15) is satisfied.

We are interested in solution candidates with $n_\infty > 0$. Solving (B.8) and (B.9) for v_Q and v_B respectively, substituting in the first-order condition (B.10) for n and taking the limit for $t \to \infty$ yields

$$q_\infty^2 + b_\infty^2 = d \tag{B.20}$$

Condition (B.20) is an *indifference condition.* It guarantees that the social planner is indifferent between all possible values for n.

Dividing by v_{Qt}, setting $t = \infty$ and rearranging, (B.13) can be written as:

$$(1/\sigma_c)\,\widehat{c}_\infty + \rho = \frac{1}{2}\mu q_\infty^{-1}\left(L_{Y\infty} + \frac{1}{\varphi}\left(\frac{X}{Q}\right)_\infty\right) + \alpha\widehat{X}_\infty + (1-\alpha)\mu n_\infty q_\infty \tag{B.21}$$

Equation (B.21) is a version of the *consumption Euler-equation,* where we replaced the shadow-prices and Lagrange-multipliers as well as their growth rates using (B.8), (B.11) and (B.6).

Both research directions, that is, increasing Q and increasing B, must yield the same social net return. We manipulate the first-order condition (B.14) for B similarly to the one for Q, using (B.9) as well as the expression $v_{St} = \left(\lambda_{Lt}\frac{1}{\varphi Q_t} - \lambda_{Yt}\alpha X_t^{\alpha-1}L_{Yt}^{1-\alpha}Q_t^{1-\alpha}\right)B_t$ from (B.7), and equations (B.11) and (B.6). Setting equal the right-hand sides of (B.21) and the modified first-order condition for B, we obtain the *research-arbitrage condition*

$$\frac{1}{2}\mu q_\infty^{-1}\left(L_{Y\infty} + \frac{1}{\varphi}\left(\frac{X}{Q}\right)_\infty\right) = \frac{1}{2}\mu b_\infty^{-1}\left(\frac{\alpha}{1-\alpha}L_{Y\infty} - \frac{1}{\varphi}\left(\frac{X}{Q}\right)_\infty\right). \tag{B.22}$$

B.3 Solution with $\widehat{S}_\infty > (-\delta)$

B.3.1 Long-run growth

In this subsection, long-run growth rates in the social optimum are derived for the case where model parameters are such that $\widehat{S}_\infty > (-\delta)$. First, we consider long-run optimal growth for parameter constellations reconcilable with condition (14) in the paper, i.e. parameter constellations for which the long-run optimal solution is characterized by deceleration. In this case, $\widehat{S}_\infty > (-\delta)$ is guaranteed by condition (15) in the paper. Afterwards, we study the case where condition (14) is not satisfied and characterize the parameter range for which $\widehat{S}_\infty > (-\delta)$ is true in this case.

ABG with deceleration ($\widehat{X}_\infty < \widehat{Q}_\infty$) If growth rates are to be constant asymptotically, equation (B.22) requires intermediate quantity in efficiency units, more precisely the ratio $(X/Q)_\infty$, to be constant in the limit as well.

A balanced growth path, along which productivity and cleanliness grow at constant rates for all t, not only asymptotically, must be characterized by a strictly positive $(X/Q)_\infty$[3]. There must therefore be equal growth in intermediate quantity, productivity and (from the resource constraint) also consumption. Further, as we consider parameter constellations with $\widehat{S}_\infty > (-\delta)$, the relation $\widehat{S}_\infty = \widehat{X}_\infty - \widehat{B}_\infty$ can be used. Equation (16G) then equals (16) in the paper and yields a ratio $\widehat{B}_\infty/\widehat{Q}_\infty$:

$$\widehat{B}_\infty/\widehat{Q}_\infty = 1 - \frac{(\sigma_c - 1)/\sigma_c}{(1-\sigma_E)/\sigma_E}. \tag{B.23}$$

If $\alpha/(1-\alpha) < 1 - \frac{(\sigma_c-1)/\sigma_c}{(1-\sigma_E)/\sigma_E}$ (see condition (14)), a balanced growth solution to the social planner's problem does not exist, because the ratio $\widehat{B}_\infty/\widehat{Q}_\infty$ in (B.23) is not reconcilable with equation (B.22) for any nonnegative $(X/Q)_\infty$. As $X/Q < 0$ has no sensible interpretation, the optimal solution is to let X/Q converge to zero asymptotically by choosing $\widehat{X}_\infty < \widehat{Q}_\infty$. According to (B.22), the optimal ratio $\widehat{B}_\infty/\widehat{Q}_\infty$ corresponds to

$$\widehat{B}_\infty/\widehat{Q}_\infty = \frac{\alpha}{1-\alpha}. \tag{B.24}$$

With the definition of the direction of technical change, it follows straigthtforwardly that technical change is green (productivity-oriented) if and only if $\alpha > 1/2$ ($\alpha < 1/2$).

To compute the relation between the growth rates $\widehat{X}_\infty$ and $\widehat{Q}_\infty$, we use (16G), substituting $\widehat{X}_\infty - \widehat{B}_\infty = \widehat{X}_\infty - \frac{\alpha}{1-\alpha}\widehat{Q}_\infty$ for $\widehat{S}_\infty$ and $\alpha\widehat{X}_\infty + (1-\alpha)\widehat{Q}_\infty$ from the resource constraint for $\widehat{c}_\infty$. After some manipulation, we obtain:

$$\widehat{X}_\infty = \frac{1 + \left(\frac{\alpha}{1-\alpha}\right)^2 - \left(1 - \frac{(\sigma_c-1)/\sigma_c}{(1-\sigma_E)/\sigma_E} - \frac{\alpha}{1-\alpha}\right)}{1 + \frac{\alpha}{1-\alpha}\left(1 - \frac{(\sigma_c-1)/\sigma_c}{(1-\sigma_E)/\sigma_E}\right)}\widehat{Q}_\infty \tag{B.25}$$

Under condition (14) in the paper ($\frac{\alpha}{1-\alpha} < 1 - \frac{(\sigma_c-1)/\sigma_c}{(1-\sigma_E)/\sigma_E}$), it is obvious that $\widehat{X}_\infty \le \widehat{Q}_\infty$ where equality is given if and only if $\widehat{Q}_\infty = 0$.

We substitute $\widehat{X}_\infty$ from (B.25) into $\widehat{c}_\infty = \alpha\widehat{X}_\infty + (1-\alpha)\widehat{Q}_\infty$ to find $\widehat{c}_\infty$ as function of $\widehat{Q}_\infty$:

$$\widehat{c}_\infty = \frac{1 + \left(\frac{\alpha}{1-\alpha}\right)^2}{1 + \frac{\alpha}{1-\alpha}\left(1 - \frac{(\sigma_c-1)/\sigma_c}{(1-\sigma_E)/\sigma_E}\right)}\widehat{Q}_\infty \tag{B.26}$$

Given condition (14), $\widehat{c}_\infty \le \widehat{Q}_\infty$ where equality is given if and only if $\widehat{Q}_\infty = 0$. Further, note that for any $\widehat{Q}_\infty > 0$, $\widehat{X}_\infty < \widehat{c}_\infty$.

To derive $\widehat{Q}_\infty$, we need to find n_∞ and q_∞. The solution for q_∞ (and also b_∞) follows directly from the indifference equation (B.20) together with (B.24):

$$q_\infty = \left(1 + \left(\frac{\alpha}{1-\alpha}\right)^2\right)^{-1/2} d^{1/2} \tag{B.27}$$

$$b_\infty = \frac{\alpha}{1-\alpha}\left(1 + \left(\frac{\alpha}{1-\alpha}\right)^2\right)^{-1/2} d^{1/2} \tag{B.28}$$

[3] On a balanced growth path, $(X/Q)_\infty = 0$ implies $X_t/Q_t = 0$ for all t. This is only possible if $X_t = c_t = 0$ for all t which cannot be an optimal path for X because the utility function satisfies the Inada-conditions for c_t.

We determine n_∞ from the consumption Euler-equation (B.21), where we replace $L_{Y\infty}$ from the labor market constraint (B.19). Further, we take into account (B.25) and (B.26) as well as $(X/Q)_\infty = 0$. With the solution

$$n_\infty = \frac{\frac{1}{2}\mu q_\infty^{-1} L - \rho}{\left(d - \frac{(1-1/\sigma_c)\left(1+\left(\frac{\alpha}{1-\alpha}\right)^2\right)}{1+\frac{\alpha}{1-\alpha}\left(1-\frac{(\sigma_c-1)/\sigma_c}{(1-\sigma_E)/\sigma_E}\right)} q_\infty^2 \right) \mu q_\infty^{-1}}$$

and q_∞ from (B.27) we find that

$$\begin{aligned} \widehat{Q}_\infty &= \mu n_\infty q_\infty \\ &= \frac{1+\frac{\alpha}{1-\alpha}\left(1-\frac{(\sigma_c-1)/\sigma_c}{(1-\sigma_E)/\sigma_E}\right)}{1/\sigma_c+\frac{\alpha}{1-\alpha}\left(1-\frac{(\sigma_c-1)/\sigma_c}{(1-\sigma_E)/\sigma_E}\right)} \frac{1}{1+\left(\frac{\alpha}{1-\alpha}\right)^2} \left(\frac{1}{2}\left(1+\left(\frac{\alpha}{1-\alpha}\right)^2\right)^{1/2} d^{-1/2}\mu L - \rho \right). \end{aligned} \tag{B.29}$$

With (B.29), the growth rates of intermediate cleanliness, intermediate quantity and consumption (and GDP) can be found from (B.24), (B.25) and (B.26) respectively.

$$\widehat{B}_\infty = \frac{1+\frac{\alpha}{1-\alpha}\left(1-\frac{(\sigma_c-1)/\sigma_c}{(1-\sigma_E)/\sigma_E}\right)}{1/\sigma_c+\frac{\alpha}{1-\alpha}\left(1-\frac{(\sigma_c-1)/\sigma_c}{(1-\sigma_E)/\sigma_E}\right)} \frac{\frac{\alpha}{1-\alpha}}{1+\left(\frac{\alpha}{1-\alpha}\right)^2} \left(\frac{1}{2}\left(1+\left(\frac{\alpha}{1-\alpha}\right)^2\right)^{1/2} d^{-1/2}\mu L - \rho \right) \tag{B.30}$$

$$\widehat{X}_\infty = \frac{1+\frac{\alpha}{1-\alpha}^2-\left(1-\frac{(\sigma_c-1)/\sigma_c}{(1-\sigma_E)/\sigma_E}-\frac{\alpha}{1-\alpha}\right)}{1/\sigma_c+\frac{\alpha}{1-\alpha}\left(1-\frac{(\sigma_c-1)/\sigma_c}{(1-\sigma_E)/\sigma_E}\right)} \frac{1}{1+\left(\frac{\alpha}{1-\alpha}\right)^2} \left(\frac{1}{2}\left(1+\left(\frac{\alpha}{1-\alpha}\right)^2\right)^{1/2} d^{-1/2}\mu L - \rho \right) \tag{B.31}$$

$$\widehat{c}_\infty = \widehat{Y}_\infty = \frac{1}{1/\sigma_c+\frac{\alpha}{1-\alpha}\left(1-\frac{(\sigma_c-1)/\sigma_c}{(1-\sigma_E)/\sigma_E}\right)} \left(\frac{1}{2}\left(1+\left(\frac{\alpha}{1-\alpha}\right)^2\right)^{1/2} d^{-1/2}\mu L - \rho \right) \tag{B.32}$$

From (16G), the growth rate of the pollution stock is:

$$\widehat{S}_\infty = \frac{\frac{(\sigma_c-1)/\sigma_c}{(1-\sigma_E)/\sigma_E}}{1/\sigma_c+\frac{\alpha}{1-\alpha}\left(1-\frac{(\sigma_c-1)/\sigma_c}{(1-\sigma_E)/\sigma_E}\right)} \cdot \left(\frac{1}{2}\sqrt{1+\left(\frac{\alpha}{1-\alpha}\right)^2} d^{-1/2}\mu L - \rho \right). \tag{B.33}$$

Balanced growth ($\widehat{X}_\infty = \widehat{Q}_\infty$) If $\frac{\alpha}{1-\alpha} > 1 - \frac{(\sigma_c-1)/\sigma_c}{(1-\sigma_E)/\sigma_E}$ so that condition (14) is not satisfied, the ratio of green to productivity-oriented research is given by (B.23):

$$\widehat{B}_\infty / \widehat{Q}_\infty = 1 - \frac{(\sigma_c - 1)/\sigma_c}{(1-\sigma_E)/\sigma_E}$$

For convex disutility of pollution ($\sigma_E < 1/2$), $\frac{(\sigma_c-1)/\sigma_c}{(1-\sigma_E)/\sigma_E}$ is smaller than one. Therefore $1 - \frac{(\sigma_c-1)/\sigma_c}{(1-\sigma_E)/\sigma_E} > 0$ and there is green innovation in the long-run optimal solution ($\widehat{B}_\infty > 0$) whenever there is productivity growth ($\widehat{Q}_\infty > 0$).

To derive the long-run optimal growth rate of Q, we proceed in the same way as in the previous paragraph. We first use the indifference condition (B.20) with (B.23) to determine q_∞ (and thereby also b_∞):

$$\begin{aligned} q_\infty &= \left(1+\left(1-\frac{(\sigma_c-1)/\sigma_c}{(1-\sigma_E)/\sigma_E}\right)^2\right)^{-1/2} \cdot d^{1/2} \\ b_\infty &= \left(1-\frac{(\sigma_c-1)/\sigma_c}{(1-\sigma_E)/\sigma_E}\right)\left(1+\left(1-\frac{(\sigma_c-1)/\sigma_c}{(1-\sigma_E)/\sigma_E}\right)^2\right)^{-1/2} \cdot d^{1/2} \end{aligned}$$

We then solve the Euler-equation (B.21) for n_∞ once more, making use of (B.20) and $\widehat{c}_\infty = \widehat{X}_\infty = \widehat{Q}_\infty = \mu n_\infty q_\infty$.

$$n_\infty = \frac{\frac{1}{2}\mu q_\infty^{-1} L - \rho}{\left(d - \left(1 - 1/\sigma_c\right) q_\infty^2\right) \mu q_\infty^{-1}}.$$

With the solutions for n_∞ and q_∞ we find that the growth rate of productivity Q is

$$\begin{aligned} \widehat{Q}_\infty &= \mu n_\infty q_\infty \\ &= \frac{1}{1/\sigma_c + \left(1 - \frac{(\sigma_c - 1)/\sigma_c}{(1-\sigma_E)/\sigma_E}\right)^2} \left(\frac{1}{2}\left(1 + \left(1 - \frac{(\sigma_c - 1)/\sigma_c}{(1-\sigma_E)/\sigma_E}\right)^2\right)^{1/2} d^{-1/2}\mu L - \rho\right). \end{aligned} \tag{B.34}$$

The growth rate of intermediate cleanliness is given by (B.23). We derive the pollution growth rate from (16G):

$$\begin{aligned} \widehat{S}_\infty &= \frac{(\sigma_c - 1)/\sigma_c}{(1-\sigma_E)/\sigma_E} \widehat{c}_\infty \\ &= \frac{\frac{(\sigma_c - 1)/\sigma_c}{(1-\sigma_E)/\sigma_E}}{1/\sigma_c + \left(1 - \frac{(\sigma_c - 1)/\sigma_c}{(1-\sigma_E)/\sigma_E}\right)^2} \cdot \left(\frac{1}{2}\sqrt{1 + \left(1 - \frac{(\sigma_c - 1)/\sigma_c}{(1-\sigma_E)/\sigma_E}\right)^2} d^{-1/2}\mu L - \rho\right) \end{aligned} \tag{B.35}$$

B.3.2 Boundary values for the rate of time preference

It follows from (B.29) and (B.34) that $\widehat{Q}_\infty > 0$ which implies that $\widehat{c}_\infty$ is positive, if and only if:

$$\rho < \overline{\rho} := \begin{cases} \frac{1}{2}\left(1 + \left(\frac{\alpha}{1-\alpha}\right)^2\right)^{1/2} d^{-1/2}\mu L, & \frac{\alpha}{1-\alpha} < 1 - \frac{(\sigma_c-1)/\sigma_c}{(1-\sigma_E)/\sigma_E} \\ \frac{1}{2}\left(1 + \left(1 - \frac{(\sigma_c-1)/\sigma_c}{(1-\sigma_E)/\sigma_E}\right)^2\right)^{1/2} d^{-1/2}\mu L, & \frac{\alpha}{1-\alpha} > 1 - \frac{(\sigma_c-1)/\sigma_c}{(1-\sigma_E)/\sigma_E} \end{cases} \tag{B.36}$$

Note that the upper bound for ρ does not depend on the rate of natural regeneration, δ, or the weight ψ of the pollution stock in utility.
$\widehat{S}_\infty > (-\delta)$ is satisfied for any $\rho < \overline{\rho}$ whenever $\sigma_c > 1$ so that $\widehat{S}_\infty > 0$. The condition on the rate of time preference needed to ensure $\widehat{S}_\infty > (-\delta)$ for $\sigma_c < 1$ follows from (B.33) and (B.35). It is given by:

$$\rho > \rho^{\text{delta}} := \begin{cases} \frac{1}{2}\left(1 + \left(\frac{\alpha}{1-\alpha}\right)^2\right)^{1/2} d^{-1/2}\mu L - \kappa_2 \frac{(1-\sigma_E)/\sigma_E}{(1-\sigma_c)/\sigma_c}\delta, & \frac{\alpha}{1-\alpha} < 1 - \frac{(\sigma_c-1)/\sigma_c}{(1-\sigma_E)/\sigma_E},\ \sigma_c < 1 \\ \frac{1}{2}\left(1 + \left(1 - \frac{(\sigma_c-1)/\sigma_c}{(1-\sigma_E)/\sigma_E}\right)^2\right)^{1/2} d^{-1/2}\mu L - \kappa_1 \frac{(1-\sigma_E)/\sigma_E}{(1-\sigma_c)/\sigma_c}\delta, & \frac{\alpha}{1-\alpha} > 1 - \frac{(\sigma_c-1)/\sigma_c}{(1-\sigma_E)/\sigma_E},\ \sigma_c < 1 \end{cases} \tag{B.37}$$

$\kappa_1 = \frac{1}{\sigma_c} + \left(1 - \frac{(\sigma_c-1)/\sigma_c}{(1-\sigma_E)/\sigma_E}\right)^2$ and $\kappa_2 = \frac{1}{\sigma_c} + \frac{\alpha}{1-\alpha}\left(1 - \frac{(\sigma_c-1)/\sigma_c}{(1-\sigma_E)/\sigma_E}\right)$ are positive constants. Note that under condition (14) (for $\frac{\alpha}{1-\alpha} < 1 - \frac{(\sigma_c-1)/\sigma_c}{(1-\sigma_E)/\sigma_E}$), the condition corresponds to condition (15) in the paper.
The transversality conditions in (TVC) require:

$$\rho > \rho^{\text{TVC}} := \begin{cases} \frac{1}{2}\frac{1-1/\sigma_c}{1+\frac{\alpha}{1-\alpha}\left(1-\frac{(\sigma_c-1)/\sigma_c}{(1-\sigma_E)/\sigma_E}\right)}\left(1 + \left(\frac{\alpha}{1-\alpha}\right)^2\right)^{1/2} d^{-1/2}\mu L, & \frac{\alpha}{1-\alpha} < 1 - \frac{(\sigma_c-1)/\sigma_c}{(1-\sigma_E)/\sigma_E} \\ \frac{1}{2}\frac{1-1/\sigma_c}{1+\left(1-\frac{(\sigma_c-1)/\sigma_c}{(1-\sigma_E)/\sigma_E}\right)^2}\left(1 + \left(1 - \frac{(\sigma_c-1)/\sigma_c}{(1-\sigma_E)/\sigma_E}\right)^2\right)^{1/2} d^{-1/2}\mu L, & \frac{\alpha}{1-\alpha} > 1 - \frac{(\sigma_c-1)/\sigma_c}{(1-\sigma_E)/\sigma_E} \end{cases} \tag{B.38}$$

Note that if $\sigma_c < 1$, condition (B.38) is satisfied for any $\rho > 0$. From (B.37) and (B.38), we define the following lower bound for ρ:

$$\underline{\rho} := \begin{cases} \rho^{\text{delta}}, & \sigma_c < 1 \\ \rho^{\text{TVC}}, & \sigma_c > 1 \end{cases}$$

B.3.3 Proof of uniqueness

The long-run growth path derived above for $\frac{\alpha}{1-\alpha} > 1 - \frac{(\sigma_c-1)/\sigma_c}{(1-\sigma_E)/\sigma_E}$ and $\frac{\alpha}{1-\alpha} < 1 - \frac{(\sigma_c-1)/\sigma_c}{(1-\sigma_E)/\sigma_E}$ respectively satisfies all the necessary conditions, given the parameter restriction $\underline{\rho} < \rho < \overline{\rho}$. It still has to be shown that the solution is unique. The only other solution candidate which has so far been excluded by the assumption of an interior solution is a solution with $n_\infty = 0$. To prove that $n_\infty = 0$ cannot be an optimal choice for n under the parameter restriction $\rho < \overline{\rho}$, we show that, given $n_\infty = 0$ and $\rho < \overline{\rho}$, the partial derivative of the Hamiltonian-function with respect to n is positive in the limit, i.e. $\lim_{t\to\infty} \frac{\partial H}{\partial n} |_{n_\infty=0} > 0$. This condition is satisfied, if and only if

$$v_{Q\infty}\mu q_\infty Q_\infty + v_{B\infty}\mu b_\infty B_\infty > \lambda_{L\infty}\left(q_\infty^2 + b_\infty^2 + d\right). \tag{B.39}$$

Given $n_\infty = 0$, the first-order conditions (B.8) and (B.9) for q and b are always satisfied and the social planner is indifferent between any levels of q_∞ and b_∞. Because every choice of q_∞ and b_∞ must yield the same level of intertemporal welfare, any particular pair can be selected as solution. We define the limits $\lim_{n_\infty\to 0} q(n_\infty)$ and $\lim_{n_\infty\to 0} b(n_\infty)$ obtained from the first-order conditions given $n_\infty > 0$ as the solutions in this case. The limit for q can be derived by solving the Euler-equation (B.21) for q instead of n. The limit for b follows from (B.23) or (B.24) respectively. It differs between the case with deceleration and the balanced-growth case.

ABG with deceleration Substituting the labor market constraint (B.1) into the Euler-equation (B.21) and taking the limit for $n_\infty \to 0$ on both sides yields $\lim_{n_\infty\to 0} q(n_\infty) = \frac{\mu}{2}L/\rho$. Accordingly, the limit for b is $\lim_{n_\infty\to 0} b(n_\infty) = \frac{\alpha}{1-\alpha}\frac{\mu}{2}L/\rho$ from (B.24). Further, we know that $\lim_{n_\infty\to 0} \frac{X}{Q}(n_\infty) = 0$.

We then determine the values of the shadow prices $v_{Q\infty}$ and $v_{B\infty}$ for $n_\infty = 0$ from (B.13) and (B.14) with (B.7) and (B.19), taking into account that X, c, Q, B and S are constant in the long run. We obtain the expressions $v_{Q\infty} = \lambda_{L\infty} Q_\infty^{-1}\left(L_{Y\infty} + \frac{1}{\varphi}\left(\frac{X}{Q}\right)_\infty\right)\frac{1}{\rho}$ and $v_{B\infty} = \lambda_{L\infty} B_\infty^{-1}\left(\frac{\alpha}{1-\alpha} L_{Y\infty} - \frac{1}{\varphi}\left(\frac{X}{Q}\right)_\infty\right)\frac{1}{\rho}$.

Substituting $v_{Q\infty}$, $v_{B\infty}$, $\lim_{n_\infty\to 0} q(n_\infty)$, $\lim_{n_\infty\to 0} b(n_\infty)$ and $\lim_{n_\infty\to 0} \frac{X}{Q}(n_\infty)$ as well as $L_{Y\infty} = L - \frac{1}{\varphi}\left(\frac{X}{Q}\right)_\infty$ in (B.39) and simplifying yields

$$\begin{aligned} \lim_{t\to\infty} \frac{\partial H}{\partial n} &> 0 \\ \Leftrightarrow \rho &< \frac{1}{2}\left(1 + \left(\frac{\alpha}{1-\alpha}\right)^2\right)^{1/2} d^{-1/2}\mu L. \end{aligned}$$

Because $\frac{1}{2}\left(1 + \left(\frac{\alpha}{1-\alpha}\right)^2\right)^{1/2} d^{-1/2}\mu L$ is the upper limit of ρ for $\frac{\alpha}{1-\alpha} < 1 - \frac{(\sigma_c-1)/\sigma_c}{(1-\sigma_E)/\sigma_E}$, we have shown that given $\rho < \overline{\rho}$ and $\frac{\alpha}{1-\alpha} < 1 - \frac{(\sigma_c-1)/\sigma_c}{(1-\sigma_E)/\sigma_E}$, no solution to the set of necessary first-order conditions with $n_\infty = 0$ exists.

Balanced growth In the balanced-growth case, it can readily be verified from (B.21) that $\lim\limits_{n_\infty \to 0} q(n_\infty) = \frac{\mu}{2} L/\rho$ as before. The limit for b changes to $\lim\limits_{n_\infty \to 0} b(n_\infty) = \left(1 - \frac{(\sigma_c - 1)/\sigma_c}{(1-\sigma_E)/\sigma_E}\right) \frac{\mu}{2} L/\rho$. The research-arbitrage condition (B.22) requires that in the limit, X/Q equals $\lim\limits_{n_\infty \to 0} \frac{X}{Q}(n_\infty) = (1-\alpha)\,\varphi\left(\frac{\alpha}{1-\alpha} - \frac{b_\infty}{q_\infty}\right) L$.

Proceeding as in the case with deceleration, we find that $\lim\limits_{t \to \infty} \frac{\partial H}{\partial n} > 0 \Leftrightarrow \rho < \frac{1}{2}\left(1 + \left(1 - \frac{(\sigma_c-1)/\sigma_c}{(1-\sigma_E)/\sigma_E}\right)^2\right)^{1/2} d^{-1/2} \mu L.$ The right-hand side corresponds to the upper bound $\overline{\rho}$ for $\frac{\alpha}{1-\alpha} > 1 - \frac{(\sigma_c-1)/\sigma_c}{(1-\sigma_E)/\sigma_E}$. Again, $\lim\limits_{t\to\infty} \frac{\partial H}{\partial n} > 0$ proves that $n_\infty = 0$ cannot be an optimal solution in the given parameter range.

B.4 Solution with $\widehat{S}_\infty = (-\delta)$

This section presents long-run optimal growth rates in case $\sigma_c < 1$ if condition (B.37) is not satisfied, so that $\widehat{S}_\infty = (-\delta)$. As for $\widehat{S}_\infty > (-\delta)$, the optimal solution may be characterized by equal growth in c, Y, X and Q or by deceleration, depending on the parameter constellation. However, even if there is no deceleration ($\widehat{X}_\infty = \widehat{Q}_\infty$), growth is only asymptotically balanced because the pollution growth rate can only converge towards $(-\delta)$ for $t \to \infty$.

ABG with deceleration

If $(X/Q)_\infty = 0$ asymptotically, equation (B.22) again yields the ratio

$$\widehat{B}_\infty / \widehat{Q}_\infty = \frac{\alpha}{1-\alpha}$$

from (B.24). From (B.20) and $\widehat{B}_\infty/\widehat{Q}_\infty = \frac{\alpha}{1-\alpha}$, we obtain the same solutions for q and b as in (B.27) and (B.28).

With $\widehat{S}_\infty = (-\delta)$, the relation between $\widehat{X}_\infty$ and $\widehat{Q}_\infty$ differs from the one in (B.25): Substituting $\widehat{c}_\infty = \alpha \widehat{X}_\infty + (1-\alpha)\,\widehat{Q}_\infty$, $\widehat{B}_\infty = \frac{\alpha}{1-\alpha}\widehat{Q}_\infty$ as well as $\widehat{S}_\infty = (-\delta)$ in the ABG-condition (16G), we obtain the following expression for $\widehat{X}_\infty$ as function of $\widehat{Q}_\infty$:

$$\widehat{X}_\infty = \frac{1}{1-\alpha} \frac{(1-2\sigma_E)/\sigma_E}{\frac{\alpha}{1-\alpha}(1/\sigma_c)+1}\delta + \frac{1 + \left(\frac{\alpha}{1-\alpha}\right)^2 - \left(1/\sigma_c - \frac{\alpha}{1-\alpha}\right)}{\frac{\alpha}{1-\alpha}(1/\sigma_c)+1}\widehat{Q}_\infty. \tag{B.40}$$

A necessary condition for deceleration is $\frac{1+\left(\frac{\alpha}{1-\alpha}\right)^2 - \left(1/\sigma_c - \frac{\alpha}{1-\alpha}\right)}{\frac{\alpha}{1-\alpha}(1/\sigma_c)+1} < 1$ which is equivalent to $1/\sigma_c > \frac{\alpha}{1-\alpha}$. Further, because $\frac{1}{1-\alpha}\frac{(1-2\sigma_E)/\sigma_E}{\frac{\alpha}{1-\alpha}(1/\sigma_c)+1}\delta > 0$, $\widehat{Q}_\infty$ must be sufficiently large, i.e.

$$\widehat{Q}_\infty > \frac{(1-2\sigma_E)/\sigma_E}{(1/\sigma_c) - \frac{\alpha}{1-\alpha}}\delta.$$

which implies an upper bound for the rate of time preference. Note that condition (14) which is necessary and sufficient for deceleration if $\widehat{S}_\infty > (-\delta)$ is sufficient but not necessary for deceleration here: $1 - \frac{(\sigma_C-1)/\sigma_C}{(1-\sigma_E)/\sigma_E} < 1/\sigma_C$ if $\sigma_C < 1$ and the disutility of pollution is convex ($\sigma_E < 1/2$). Therefore $\frac{\alpha}{1-\alpha} < 1 - \frac{(\sigma_C-1)/\sigma_C}{(1-\sigma_E)/\sigma_E}$ is a stricter condition than $\frac{\alpha}{1-\alpha} < 1/\sigma_C$. Further, the condition $\widehat{S}_\infty = (-\delta)$ can, because of (16G), be expressed as a condition on $\widehat{c}_\infty$, i.e. $\widehat{c}_\infty \geq \frac{(1-\sigma_E)/\sigma_E}{(1-\sigma_C)/\sigma_C}\delta$. Given $\frac{\alpha}{1-\alpha} < 1/\sigma_C$ and $\frac{\alpha}{1-\alpha} < 1 - \frac{(\sigma_C-1)/\sigma_C}{(1-\sigma_E)/\sigma_E}$, the condition for $\widehat{S}_\infty$ to converge to $(-\delta)$, which is $\widehat{c}_\infty \geq \frac{(1-\sigma_E)/\sigma_E}{(1-\sigma_C)/\sigma_C}\delta$, implies $\widehat{c}_\infty > \frac{(1-2\sigma_E)/\sigma_E}{(1/\sigma_C) - \frac{\alpha}{1-\alpha}}\delta$ which in turn, as $\widehat{c}_\infty \leq \widehat{Q}_\infty$, guarantees that $\widehat{Q}_\infty > \frac{(1-2\sigma_E)/\sigma_E}{(1/\sigma_C) - \frac{\alpha}{1-\alpha}}\delta$.

With $\widehat{X}_\infty$ from (B.40), $\widehat{c}_\infty = \alpha \widehat{X}_\infty + (1-\alpha)\widehat{Q}_\infty$ yields

$$\widehat{c}_\infty = \frac{\alpha}{1-\alpha}\frac{(1-2\sigma_E)/\sigma_E}{\frac{\alpha}{1-\alpha}(1/\sigma_c)+1}\delta + \frac{1+\left(\frac{\alpha}{1-\alpha}\right)^2}{\frac{\alpha}{1-\alpha}(1/\sigma_c)+1}\widehat{Q}_\infty. \tag{B.41}$$

In the same way as before, we derive n_∞ from the Euler-equation. The solution is:

$$n_\infty = \frac{\frac{1}{2}\mu q_\infty^{-1} L - \rho + (1-1/\sigma_c)\frac{\alpha}{1-\alpha}\frac{(1-2\sigma_E)/\sigma_E}{\frac{\alpha}{1-\alpha}(1/\sigma_c)+1}\delta}{\left(d - \frac{(1-1/\sigma_c)\left(1+\left(\frac{\alpha}{1-\alpha}\right)^2\right)}{\frac{\alpha}{1-\alpha}(1/\sigma_c)+1}q_\infty^2\right)\mu q_\infty^{-1}}$$

And the productivity growth rate is given by:

$$\begin{aligned}
\widehat{Q}_\infty &= \mu n_\infty q_\infty \\
&= (1-\alpha)\left(\frac{\alpha}{1-\alpha}+\sigma_c\right)\left(1+\left(\frac{\alpha}{1-\alpha}\right)^2\right)^{-1} \\
&\cdot\left(\frac{1}{2}\left(1+\left(\frac{\alpha}{1-\alpha}\right)^2\right)^{1/2} d^{-1/2}\mu L - \rho + (1-1/\sigma_c)\frac{\alpha}{1-\alpha}\frac{(1-2\sigma_E)/\sigma_E}{\frac{\alpha}{1-\alpha}(1/\sigma_c)+1}\delta\right).
\end{aligned} \tag{B.42}$$

The growth rate of intermediate cleanliness is the same as in (B.30). The growth rates of intermediate quantity and consumption (and GDP) follow from (B.40) and (B.41).

$$\begin{aligned}
\widehat{X}_\infty &= \frac{1}{1-\alpha}\frac{(1-2\sigma_E)/\sigma_E}{\frac{\alpha}{1-\alpha}(1/\sigma_c)+1}\delta + (1-\alpha)\sigma_c\frac{1+\left(\frac{\alpha}{1-\alpha}\right)^2-\left(1/\sigma_c-\frac{\alpha}{1-\alpha}\right)}{1+\left(\frac{\alpha}{1-\alpha}\right)^2} \\
&\cdot\left(\frac{1}{2}\left(1+\left(\frac{\alpha}{1-\alpha}\right)^2\right)^{1/2} d^{-1/2}\mu L - \rho + (1-1/\sigma_c)\frac{\alpha}{1-\alpha}\frac{(1-2\sigma_E)/\sigma_E}{\frac{\alpha}{1-\alpha}(1/\sigma_c)+1}\delta\right)
\end{aligned} \tag{B.43}$$

$$\begin{aligned}
\widehat{c}_\infty &= \widehat{Y}_\infty = \frac{\alpha}{1-\alpha}\frac{(1-2\sigma_E)/\sigma_E}{\frac{\alpha}{1-\alpha}(1/\sigma_c)+1}\delta + (1-\alpha)\sigma_c \\
&\cdot\left(\frac{1}{2}\left(1+\left(\frac{\alpha}{1-\alpha}\right)^2\right)^{1/2} d^{-1/2}\mu L - \rho + (1-1/\sigma_c)\frac{\alpha}{1-\alpha}\frac{(1-2\sigma_E)/\sigma_E}{\frac{\alpha}{1-\alpha}(1/\sigma_c)+1}\delta\right)
\end{aligned} \tag{B.44}$$

ABG without deceleration

Without deceleration, it still holds that $\widehat{c}_\infty = \widehat{X}_\infty = \widehat{Q}_\infty$. Substituting $\widehat{c}_\infty = \widehat{X}_\infty = \widehat{Q}_\infty$ into the ABG-condition (16G) with $\widehat{S}_\infty = (-\delta)$ and solving for $\widehat{B}_\infty$, we obtain

$$\widehat{B}_\infty = (1/\sigma_c)\widehat{Q}_\infty - ((1-2\sigma_E)/\sigma_E)\delta. \tag{B.45}$$

As $\sigma_c < 1$, the ratio $\widehat{B}_\infty/\widehat{Q}_\infty$ in (B.45) is larger than $\widehat{B}_\infty/\widehat{Q}_\infty = 1 - \frac{(\sigma_c-1)/\sigma_c}{(1-\sigma_E)/\sigma_E}$ in (B.23) for every constellation of parameters that satisfies the condition $\widehat{Q}_\infty (= \widehat{c}_\infty) > \frac{(1-\sigma_E)/\sigma_E}{(1-\sigma_c)/\sigma_c}\delta$ for $\widehat{S}_\infty = (-\delta)$. It can be verified that $\widehat{B}_\infty/\widehat{Q}_\infty < \alpha/(1-\alpha)$, so that the research arbitrage equation is satisfied, if either $(1/\sigma_c) < \frac{\alpha}{1-\alpha}$ or if $(1/\sigma_c) > \frac{\alpha}{1-\alpha}$ and at the same time $\widehat{Q}_\infty > \frac{(1-2\sigma_E)/\sigma_E}{(1/\sigma_c)-\frac{\alpha}{1-\alpha}}\delta$.

Using the indifference condition (B.20), $q_\infty^2 + b_\infty^2 = d$, to express b_∞ as function of q_∞, equation (B.45) can also be written as

$$\mu n_\infty \sqrt{d - q_\infty^2} = (1/\sigma_c)\, \mu n_\infty q_\infty - ((1 - 2\sigma_E)/\sigma_E)\, \delta \tag{B.46}$$

The consumption Euler-equation (B.21) does not depend on $\widehat{S}_\infty$ or the ratio $\widehat{B}_\infty / \widehat{Q}_\infty$. It yields the same relation

$$n_\infty = \frac{\frac{1}{2}\mu q_\infty^{-1} L - \rho}{\left(d - (1 - 1/\sigma_c)\, q_\infty^2\right) \mu} q_\infty \tag{B.47}$$

between n_∞ and q_∞ as in the case without deceleration and $\widehat{S}_\infty > (-\delta)$.

Equations (B.46) and (B.47) form a system of two equations in the two unknowns q_∞ and n_∞. However, after substituting (B.47) for n_∞ in (B.46), it is not possible to solve (B.46) for q_∞ analytically due to the mixture of exponents.

Depending on parameters, there may be a unique solution, two solutions or no solution. To prove this claim, consider equation (B.46), where $n_\infty = n_\infty(q_\infty)$ is given by (B.47). We divide both sides of equation (B.46) by $n_\infty(q_\infty)$:

$$\mu \sqrt{d - q_\infty^2} = (1/\sigma_c)\, \mu q_\infty - n_\infty^{-1}(q_\infty)\, ((1 - 2\sigma_E)/\sigma_E)\, \delta$$

The left-hand side of the modified equation is non-negative as well as decreasing and concave in q_∞. The right-hand side is positive whenever the condition for $\widehat{S}_\infty = (-\delta)$ is satisfied, because $\mu n_\infty q_\infty = \widehat{Q}_\infty > \frac{(1-2\sigma_E)/\sigma_E}{1/\sigma_c}\delta$ is a weaker condition than $\widehat{Q}_\infty > \frac{(1-\sigma_E)/\sigma_E}{(1-\sigma_c)/\sigma_c}\delta$. For $\sigma_c < 1$, in the relevant range with $\widehat{Q}_\infty > \frac{(1-\sigma_E)/\sigma_E}{(1-\sigma_c)/\sigma_c}\delta$, the right-hand side is concave and first increasing, then decreasing in q_∞ because the first term is linear and $n_\infty^{-1}(q_\infty)$ is decreasing and convex in q_∞ whenever $\sigma_c < 1$.

A unique solution exists if and only if the value $q_\infty = \sqrt{d}$, which sets the left-hand side of the equation to zero, lies between the two zeros of the right-hand side. An equivalent condition is that at $q_\infty = \sqrt{d}$, the right-hand side is positive. This is true if and only if

$$\rho < \frac{1}{2}\mu d^{-1/2} L - ((1 - 2\sigma_E)/\sigma_E)\, \delta.$$

B.5 Proof of corollary 2

B.5.1 Solution

Without the pollution externality, utility depends on consumption only:

$$U = \int_{t=0}^{\infty} e^{-\rho t} \frac{\sigma_c}{\sigma_c - 1} c_t^{\frac{\sigma_c - 1}{\sigma_c}} L dt$$

The first-order condition for S becomes

$$\frac{\partial H}{\partial S_t} = \rho v_{St} - \dot{v}_{St} \Leftrightarrow -\delta v_{St} = \rho v_{St} - \dot{v}_{St}. \tag{B.48}$$

This condition can only be satisfied if $v_{St} = 0$ and $\dot{v}_{St} = 0$ for all t. The second solution $\widehat{v}_{1t} = \rho + \delta$ violates the transversality condition for S for all possible long-run growth rates $\widehat{S}_\infty \geq -\delta$.

The first-order condition (B.7) for X_t then directly yields aggregate intermediate production

$$X_t = \frac{\alpha}{(1 - \alpha)} \varphi Q_t L_{Yt} \tag{B.49}$$

for any given labor supply L_{Yt} and productivity level Q_t.
With $v_{St} = 0$, it follows from the first-order condition

$$v_{Bt}\mu n_t b_t = \rho v_{Bt} - \dot{v}_{Bt}$$

for B_t that $v_{Bt} = 0$ and $\dot{v}_{Bt} = 0$ for all t[4].
If $v_{Bt} = 0$, it is optimal to set $b_t = 0$ for all t as can be seen from (B.9). Then the optimal long-run level of q is

$$q_\infty^{\psi=0} = \sqrt{d} \tag{B.50}$$

from (B.20).

As $L_{Y\infty}$ is constant, we conclude from (B.49) and the resource constraint that $\widehat{X}_\infty = \widehat{c}_\infty = \widehat{Y}_\infty = \widehat{Q}_\infty$. We can still determine n_∞ from (B.21) using $\widehat{X}_\infty = \widehat{c}_\infty = \widehat{Q}_\infty$, $q_\infty^{\psi=0} = \sqrt{d}$, the labor market constraint (B.19) and (B.20):

$$n_\infty^{\psi=0} = \frac{\sigma_c}{\sqrt{d}\mu}\left(\frac{1}{2}\mu d^{-1/2}L - \rho\right)$$

The consumption growth rate of the economy is:

$$\begin{aligned} \widehat{c}_\infty^{\psi=0} &= \widehat{Q}_\infty^{\psi=0} = \mu n_\infty q_\infty \\ &= \frac{1}{1/\sigma_c}\left(\frac{1}{2}\mu d^{-1/2}L - \rho\right) \end{aligned} \tag{B.51}$$

B.5.2 Comparison to the solution in theorem 1

(i) **Parameter restriction for positive long-run growth:** Given conditions (14) and (15), the upper bound for ρ in the baseline model with $\psi > 0$ is $\overline{\rho} = \frac{1}{2}\mu\left(1 + \left(\frac{\alpha}{1-\alpha}\right)^2\right)^{1/2} d^{-1/2}L$. From (B.51), it follows that the upper bound for positive long-run growth with $\psi = 0$ is $\overline{\rho}^{\psi=0} = \frac{1}{2}\mu d^{-1/2}L$. Comparison of $\overline{\rho}$ and $\overline{\rho}^{\psi=0}$ shows that positive consumption growth is optimal for larger values of the rate of time preference when $\psi > 0$ because $\left(1 + \left(\frac{\alpha}{1-\alpha}\right)^2\right)^{1/2} > 1$.

(ii) **Comparison of $\widehat{c}_\infty$:** Given conditions (14) and (15), the long-run consumption growth rate in the baseline model is given by (B.32). Comparison of (B.32) to the growth rate in (B.51) proves the claim in the theorem.

(iii) **Influence of the size of ψ:** From the previous subsections, it is obvious that long-run growth rates are not affected by the parameter ψ.

Note that the validity of results is not limited to the parameter range with deceleration and $\widehat{S}_\infty > (-\delta)$ which is considered in the theorem.

[4] Again, there is a second solution, $\widehat{v}_{Bt} = \rho - \mu n_t b_t$. However, like the non-zero solution for v_{St}, it does not satisfy the transversality condition for the associated state-variable (B).

C Appendix to section 5: The model with a non-renewable resource

C.1 The laissez-faire equilibrium

C.1.1 The representative household

As we assume that the representative household owns the resource stock, the budget constraint in period t becomes

$$C_t + \dot{A}_t = r_t A_t + w_{Yt} L_{Yt} + w_{Dt} L_{Dt} + p_{Rt} R_t.$$

The household maximizes intertemporal utility subject to the budget constraint and the natural resource constraint (18). Denote the Lagrange-multiplier for the natural resource constraint by λ_{Rt}. Two new conditions are added to the set of necessary first-order conditions:

$$\frac{\partial H}{\partial R_t} = 0 \Leftrightarrow v_{At} p_{Rt} = \lambda_{Rt} \tag{C.1}$$

$$\frac{\partial H}{\partial \lambda_{Rt}} \leq 0 \Leftrightarrow \int_0^\infty R_t dt \leq F_0 \quad \lambda_{Rt} \geq 0 \quad \lambda_{Rt} \left(F_0 - \int_0^\infty R_t dt \right) = 0 \tag{C.2}$$

The new first-order condition (C.2) for λ_{Rt} together with the first-order condition (C.1) for R_t states that either the resource stock is fully exhausted asymptotically, or the price p_{Rt} of the resource must be zero for all t: If $\int_0^\infty R_t dt < F_0$, λ_{Rt} must be zero by the complementary-slackness condition $\lambda_{Rt} \left(F_0 - \int_0^\infty R_t dt \right) = 0$. But if $\lambda_{Rt} = 0$ for some $t < \infty$, then $\lambda_{Rt} = 0$ for all $t < \infty$ since $\lambda_{Rt} = 0$ can only be satisfied if $\widetilde{\lambda}_R = 0$. By (C.1), it follows that $p_{Rt} = 0$ for all t. We conclude that in the laissez-faire equilibrium, the resource stock is always fully depleted asymptotically and $\lambda_{Rt} > 0$.
It can be shown that the standard consumption Euler-equation (A.6),

$$\widehat{c}_t = \sigma_c \cdot (r_t - \rho),$$

still holds.
Using $\widehat{\lambda}_R = \rho$, the first-order condition for R_t together with the first-order-condition for A_t (equation (A.4)) yields the Hotelling rule:

$$\begin{aligned} \widehat{p}_{Rt} &= \widehat{\lambda}_R - \widehat{v}_{At} \\ &= \rho - (\rho - r_t) \\ &= r_t \end{aligned} \tag{C.3}$$

C.1.2 Production

The demand function (A.8) for intermediate goods remains unchanged vis-à-vis the baseline model:

$$X_{it}^d(p_{it}, L_{Yt}, Q_{it}) = \left(\frac{\alpha}{p_{it}} \right)^{\frac{1}{1-\alpha}} Q_{it} L_{Yt}$$

In the profit function

$$\pi_{it}^X = (p_{it} - MC_t) X_{it},$$

it has to be taken into account that marginal production costs correspond to the price p_{Rt} for the resource instead of marginal labor costs. Marginal costs are still the same for every firm j so that again, only the firm with the latest patent will be active in production. The profit-maximizing monopoly price, given by the constant mark-up $\frac{1}{\alpha}$ over marginal costs, is

$$p_{it} = p_t = \frac{1}{\alpha} \cdot p_{Rt} \tag{C.4}$$

C.1.3 Resource market clearing

Because the resource stock is fully exhausted, total resource demand $\int_0^\infty R_t dt = \int_0^\infty X_t^d dt$ must equal total supply F_0. Integrating (A.8) over all sectors i and using the Hotelling-rule to describe the development of the resource price, the condition can be written as

$$\left(\frac{\alpha^2}{p_{R0}}\right)^{\frac{1}{1-\alpha}} \int_0^\infty e^{-\frac{1}{1-\alpha}\int_0^t r_v dv} L_{Yt} Q_t dt = F_0. \tag{C.5}$$

Given the paths for productivity Q, labor L_Y and the interest rate, condition (C.5) fixes the resource price at $t = 0$ for any given initial resource stock F_0. It thereby determines the level of the path $\{p_{Rt}\}_0^\infty$. The more resource-abundant the economy is, the smaller is the resource price in every period t.

C.1.4 Proposition C.1: Effects of resource scarcity on the laissez-faire equilibrium

The resource price increases over time according to the Hotelling rule (C.3). Through the increasing resource price, resource scarcity leads to a restriction in the growth rate of intermediate quantity below productivity growth along the balanced growth path. More precisely, intermediate quantity must fall over time according to lemma 3 in the paper. The increasing resource price leads to persistent quantity degrowth. While this is obvious, it is interesting to study the impact of resource scarcity on the laissez-faire equilibrium more extensively.

For any size of the finite initial resource stock F_0, the fact that the resource becomes scarcer over time (the finiteness of the resource) is reflected in the positive growth rate of the resource price. Further, resource scarcity affects economic variables through the size of the initial stock F_0, which has a negative influence on the level of the resource price (see equation (C.9)).

While the size of F_0 is exogenous, the growth rate of the resource price is endogenous. Still, it is possible to single out the effects not only of the size of F_0 but also the impact of the growing resource price on equilibrium growth and the levels of technology, production, consumption and pollution:

Proposition C.1 ***Effects of resource scarcity on the laissez-faire equilibrium***
(a) Growth effects: *The introduction of a finite initial resource stock $F_0 < \infty$, through an increasing resource price, (i) leads to persistent quantity degrowth (ii) unambiguously decreases the equilibrium growth rates of Y and c as well as the long-run growth rate of S, (iii) lowers (increases) the productivity growth rate whenever $\sigma_c > 1$ ($\sigma_c < 1$) (given it is positive), and (iv) restricts the parameter range for which there is positive growth in per capita consumption.*
(b) Level effects: *The size of the initial resource stock F_0 (i) does not affect the paths of the technology stocks Q and B. However, the smaller F_0, (ii) the larger is the level of the resource price along the entire equilibrium path*

and (iii) the smaller are, accordingly, the levels of intermediate production, output and per capita consumption in every period t and the smaller the pollution level in every period $t > 0$.

Proof. See appendix C.1.5. ■

The growth rate of the resource price has two opposing effects on the productivity growth rate, $\widehat{Q}^{\mathrm{LF,R}}$: On the one hand, the growing price depresses monopoly profits from intermediate production and thereby entry for a given interest rate. This tends to slow growth. On the other hand, the decrease in entry causes a countervailing general equilibrium effect: The equilibrium interest rate is smaller, which slows the price increase, as can be seen from (C.3), and stimulates entry and productivity growth.

If $\sigma_c < 1$, the representative household desires to smooth consumption over time and reacts inelastically to changes in the interest rate. The decline in the interest rate is therefore more pronounced than for $\sigma_c > 1$. This explains why the positive effect on productivity growth predominates in the former, and the negative effect predominates in the latter case.

The growing resource price depresses consumption growth along the equilibrium path because it induces quantity degrowth. For $\sigma_c < 1$, the increase in productivity growth has a countervailing positive effect, both directly and because it dampens the decline in intermediate quantity. Nevertheless, the overall effect of resource scarcity on consumption and output growth is unambiguously negative for any value of σ_c.[5]

On the other hand, resource scarcity has a beneficial effect on household utility through the pollution externality: The growing resource price ensures that the total amount of emissions at equilibrium is bounded and the pollution stock declines along the (asymptotically) balanced growth path.

The initial resource stock affects the laissez-faire equilibrium only through the level of the resource price. The price level leaves growth rates unaffected. The reason is that the level of the resource price does not influence research profits because the return to and the costs of R&D decline in the price level in the same way. It follows that two economies with different initial resource endowments share the same long-run growth rates and the same technology paths.

On the other hand, the price level is relevant for the determination of intermediate production levels in each period. The higher the resource price, the higher is the price firms in the consumption goods sector pay for intermediate goods. The lower are therefore intermediate demand and the equilibrium quantity of intermediates. Productivity and labor in the consumption goods sector are independent of the initial resource stock. Accordingly, the paths for output and consumption in an economy with small initial resource stock are below those of a more resource-abundant economy. At the same time, there is less pollution in every period as less of the polluting input is produced.

C.1.5 Proof of proposition C.1

(a) Growth effects

(i) **Quantity degrowth:** Along the lines of the baseline model, it can be shown that the economy adjusts to its balanced-growth path without transitional dynamics. The allocation of labor is constant for all t.

[5] Contrary to the equilibrium in the baseline model, it is even possible that the growth rates of consumption and output are negative (see also Schou (2002) for a similar result). Degrowth does then not only occur in polluting quantity but in consumption and output.

Integrating (A.8) over i, computing the growth rate and using (C.3) then yields

$$\widehat{X}^{\text{LF, R}} = \widehat{Q}^{\text{LF, R}} - \frac{1}{1-\alpha} r^{\text{LF,R}}. \tag{C.6}$$

The transversality condition for assets requires $r^{\text{LF,R}} > \widehat{Q}^{\text{LF,R}}$. As $\frac{1}{1-\alpha} > 1$, this is sufficient for $\widehat{X}^{\text{LF,R}}$ in (C.6) to be negative.

(ii) **Growth rates of** c, Y, S: From the Euler-equation, taking into account $\widehat{Y} = \widehat{c}$, it follows that the introduction of a finite resource stock $F_0 < \infty$ lowers output and consumption growth if and only if it decreases the equilibrium interest rate. The equilibrium interest rate can be found to be

$$r^{\text{LF, R}} = \frac{\frac{1}{2}\frac{1}{\sigma_c}\mu L\left(\sqrt{1+d}-1\right) + \left(\frac{1-\alpha}{\alpha}\left(\sqrt{1+d}-1\right)+d\right)\rho}{\frac{1}{\alpha}\frac{1}{\sigma_c}\left(\sqrt{1+d}-1\right)^2 + \frac{1-\alpha}{\alpha}\left(\sqrt{1+d}-1\right) + d + \kappa_{R,1}}, \tag{C.7}$$

for all t, where $\kappa_{R,1} := \frac{\alpha}{1-\alpha}\frac{1}{\sigma_c}\left(\frac{1}{\alpha}\left(\sqrt{1+d}-1\right)^2 + \frac{1-\alpha}{\alpha}\left(\sqrt{1+d}-1\right)+d\right)$. The term $\kappa_{R,1} > 0$ in the denominator is only present because with $F_0 < \infty$, the resource price p_{R_t} increases over time according to the Hotelling rule (C.3). The introduction of a finite initial resource stock decreases the equilibrium interest rate and therefore growth in output and consumption.

The effect of the finiteness of the resource stock on the equilibrium growth rate of S is obvious, as the long-run pollution growth rate is negative precisely because the resource stock is exhaustible and the resource price increases.

(iii) **Growth rate of** Q: Setting equal the consumption Euler-equation and the relation $\widehat{c} = \alpha\widehat{X} + (1-\alpha)\widehat{Q}$ where $\widehat{X}$ is given by (C.6), $\widehat{Q}^{\text{LF,R}}$ can be determined. After some manipulation, the growth rate

$$\widehat{Q}^{\text{LF,R}} = \frac{\frac{1}{2}\mu L - \frac{1}{\alpha}\left(\sqrt{1+d}-1\right)\rho + k_{R,2}}{\frac{1}{\alpha}\frac{1}{\sigma_c}\left(\sqrt{1+d}-1\right)^2 + \frac{1-\alpha}{\alpha}\left(\sqrt{1+d}-1\right) + d + \kappa_{R,1}}\left(\sqrt{1+d}-1\right), \tag{C.8}$$

is derived, with $\kappa_{R,2} = \frac{\alpha}{1-\alpha}\left(\frac{1}{\sigma_c}\frac{1}{2}\mu L - \frac{1}{\alpha}\left(\sqrt{1+d}-1\right)\rho\right)$. Both $\kappa_{R,1}$ and $\kappa_{R,2}$ are attributable to the positive growth rate of the resource price associated with the introduction of an exhaustible resource stock $F_0 < \infty$. Setting $\kappa_{R,1}$ and $\kappa_{R,2}$ to zero and comparing the resulting expression to $\widehat{Q}^{\text{LF,R}}$ proves that the growing resource price decreases the productivity growth rate if and only if

$$\begin{aligned} k_{R,2}&\left(\frac{1}{\alpha}\frac{1}{\sigma_c}\left(\sqrt{1+d}-1\right)^2 + \frac{1-\alpha}{\alpha}\left(\sqrt{1+d}-1\right)+d\right) \\ &< \left(\frac{1}{2}\mu L - \frac{1}{\alpha}\left(\sqrt{1+d}-1\right)\rho\right)\kappa_{R,1} \qquad . \end{aligned}$$

Substituting the expressions for $\kappa_{R,1}$ and $k_{R,2}$, the condition becomes

$$\frac{1}{\alpha}\frac{1-\sigma_c}{\sigma_c}\left(\sqrt{1+d}-1\right)\left(\frac{1}{2}\frac{1}{\sigma_c}\mu L\left(\sqrt{1+d}-1\right) + \left(\frac{1-\alpha}{\alpha}\left(\sqrt{1+d}-1\right)+d\right)\rho\right) < 0$$

which is equivalent to $\sigma_c > 1$. For $\sigma_c < 1$, the growing resource price increases the productivity growth rate.

(iv) **Parameter restriction for positive long-run growth:** Substituting the solution for $r^{\text{LF, R}}$ in the Euler-equation, the upper bound on ρ which guarantees positive consumption growth can be shown to equal $\overline{\rho}_{\rm c}^{\rm LF,R} = \frac{1}{2}\mu L \frac{1}{\frac{1}{\alpha}\left(\sqrt{1+d}-1\right)+\kappa_{R,3}}$. The expression $\kappa_{R,3}$, defined as $\kappa_{R,3} := \frac{1}{1-\alpha}\left(\sqrt{1+d}-1\right) + \frac{1}{\alpha^2} + d/\left(\sqrt{1+d}-1\right)$, results from the growth rate of the resource price. As $\kappa_{R,3} > 0$, the increasing resource price lowers the upper bound on ρ.

(b) Level effects

(i) **Technology path:** It is obvious from the previous paragraph along with $\widehat{B}^{\rm LF,R} = 0$ that the initial resource stock F_0 does not influence the growth rates of Q and B along the equilibrium path. Because the initial values for Q and B are given and the growth rates of Q and B jump to their respective ABG-levels directly, it follows that the entire paths of Q and B do not depend on F_0.

(ii) **Level of p_R:** Taking into account that $r = r^{\rm LF,R}$ and L_Y are constant along the equilibrium path and Q grows at the constant rate $\widehat{Q}^{\rm LF,R}$ for all t, solving equation (C.5) for the resource price in $t = 0$ yields

$$p_{R0} = \alpha^2 \left(\frac{Q_0 L_Y}{F_0}\right)^{1-\alpha} \left(\frac{1}{1-\alpha} r^{\rm LF,R} - \widehat{Q}^{\rm LF,R}\right)^{1-\alpha}, \tag{C.9}$$

with $r^{\rm LF,R}$ and $\widehat{Q}^{\rm LF,R}$ given by (C.7) and (C.8). Using the Hotelling-rule (C.3), the resource price can be determined at any point in time. A decline in F_0 increases the price for all t.

(iii) **Levels of** X, c, Y, S: It has been shown in (i) that the path for Q is unaffected by a variation in F_0. It can be shown that the same is true for the constant L_Y. Further, there is no direct effect of F_0 on X, c, Y and S. However, intermediate demand in every period t decreases in the resource price according to equation (A.8) and (C.4). It follows that by increasing the resource price for all t, a decline in F_0 shifts the path for intermediate quantity downwards.

Because $\{L_Y\}_0^\infty$ and $\{Q_t\}_0^\infty$ are independent of F_0, the path for output and consumption shifts downwards with the path for X.

Further, because $\{B_t\}_0^\infty$ is not affected by F_0, emissions X_t/B_t are lower for all t. The path for the pollution stock S_t is given by the solution to the differential equation (8), $\dot{S}_t = X_t/B_t - \delta S_t$. From the general solution, it can be concluded that due to the decline in emissions, the pollution stock S is lower in every period $t > 0$.

C.2 Appendix to section 5.2: Resource scarcity in the long-run social optimum

C.2.1 First-order conditions

Three changes occur in the set of necessary first-order conditions compared to the baseline model: First, instead of the marginal labor requirement, the shadow price λ_R of the non-renewable resource contributes to the marginal social cost of intermediate production so that the first-order condition for X becomes

$$\frac{\partial H}{\partial X_t} = 0 \Leftrightarrow \frac{v_{St}}{B_t} + \lambda_{Yt} \alpha X_t^{\alpha-1} L_{Yt}^{1-\alpha} Q_t^{1-\alpha} - \lambda_{Rt} = 0 \tag{C.10}$$

In the first-order condition (B.13) for Q, the last term on the left-hand side $(\lambda_{Lt}(1/\varphi)\left(X_t/Q_t^2\right)$ drops out because Q no longer affects the production of intermediate goods.
Second, the first-order conditions are complemented by a complementary slackness condition:

$$\frac{\partial H}{\partial \lambda_{Rt}} \leq 0 \Leftrightarrow F_0 - \int_0^\infty X_t dt \geq 0 \quad \lambda_{Rt} \geq 0 \quad \lambda_{Rt}\left(F_0 - \int_0^\infty X_t dt\right) = 0 \tag{C.11}$$

Third, labor is only allocated to research and output production which changes the first order condition for λ_{Lt} to:

$$\frac{\partial H}{\partial \lambda_{Lt}} = 0 \Leftrightarrow L = L_{Yt} + n_t(q_t^2 + b_t^2 + d) \tag{C.12}$$

The set of first-order conditions is otherwise unaffected by the modifications in the model setup.

C.2.2 Proof of proposition 2

(a) Binding constraint

(i) **Quantity degrowth:** If there is quantity degrowth, $\widehat{S}_\infty = 0$ so that $v_{S\infty} = 0$, while λ_R grows persistently. To satisfy the first-order condition (C.10) for X, the social marginal product of X in production must equal λ_R asymptotically:

$$c_\infty^{-1/\sigma_c} \alpha X_\infty^{\alpha-1} L_{Y\infty}^{1-\alpha} Q_\infty^{1-\alpha} = \lambda_{R\infty} \tag{C.13}$$

Note that we already substituted $\lambda_Y = c_\infty^{-1/\sigma_c}$ from the first-order condition for c. Condition (C.13) replaces condition (16) for asymptotically-balanced growth from the baseline model. Computing growth rates on both sides of (C.13) yields $(-1/\sigma_c \cdot \widehat{c}_\infty) - (1-\alpha)\left(\widehat{X}_\infty - \widehat{Q}_\infty\right) = \rho$. From this equation, using $\widehat{c}_\infty = \alpha \widehat{X}_\infty + (1-\alpha)\widehat{Q}_\infty$, we derive the growth rate $\widehat{X}_\infty^{\mathrm{R}}$ for any given $\widehat{Q}_\infty^{\mathrm{R}}$:

$$\widehat{X}_\infty^{\mathrm{R}} = \frac{1}{\frac{\alpha}{1-\alpha}\frac{1}{\sigma_c} + 1}\left(\left(1 - \frac{1}{\sigma_c}\right)\widehat{Q}_\infty^{\mathrm{R}} - \frac{1}{1-\alpha}\rho\right) \tag{C.14}$$

If $\sigma_c < 1$, it can be seen directly that $\widehat{X}_\infty^{\mathrm{R}} < 0$. For $\sigma_c > 1$ the transversality conditions, which require $\rho > \left(1 - \frac{1}{\sigma_c}\right)\widehat{Q}_\infty^{\mathrm{R}}$, together with $(1-\alpha) < 1$ guarantee that indeed $\widehat{X}_\infty^{\mathrm{R}} < 0$.

(ii) **Green Innovation:** The research-arbitrage equation is:

$$\frac{\mu}{2q_\infty} L_{Y\infty} = \frac{\mu}{2b_\infty} L_{Y\infty}\left(\frac{\alpha}{1-\alpha} - \frac{1}{1-\alpha}\left(\frac{\lambda_R}{\lambda_Y}\right)_\infty \left(\frac{X}{Q}\right)_\infty^{1-\alpha} L_{Y\infty}^{\alpha-1}\right) \tag{C.15}$$

Substituting (C.13) in (C.15) shows that investing in the cleanliness of technology is not optimal in the long run:

$$\begin{aligned} \frac{\mu}{2b_\infty} L_{Y\infty}\left(-\frac{\alpha}{1-\alpha} + \frac{\alpha}{1-\alpha}\right) &= (\rho - (1 - 1/\sigma_c)\,\widehat{c}_\infty) \\ \Leftrightarrow b_\infty^{\mathrm{R}} &= 0 \end{aligned}$$

From $q_\infty^2 + b_\infty^2 = d$ it follows that $q_\infty^{\mathrm{R}} = \sqrt{d}$ so that labor in the R&D-sector is entirely used for productivity-oriented innovation.

(b) Unbinding constraint

(i) **Convergence of $\int_0^\infty X_t dt$:** The integral $\int_0^\infty X_t dt$ can be written as the sum of the two integrals $\int_0^T X_t dt$ and $\int_T^\infty X_t dt$. It converges if and only if both integrals in the sum converge.

Because X_t is finite for every t, the definite integral $\int_0^T X_t dt$ assumes a finite value.

Consider the second integral: In any solution to the social planner's problem for which growth rates converge to the growth rates of the asymptotically-balanced growth solution with quantity degrowth from section 4.3 in the paper, the sequence $\left\{\widehat{X}_t\right\}_0^\infty$ converges to the constant $\widehat{X}_\infty < 0$. Assuming continuity, convergence implies that there exists a time T such that $\widehat{X}_t < \overline{\widehat{X}} < 0$ for all $t > T$. Therefore, if the integral $\int_T^\infty X_T e^{\overline{\widehat{X}} \cdot t} dt$ converges, so does the integral $\int_T^\infty X_t dt$. The limit of the integral $\int_T^\infty X_T e^{\overline{\widehat{X}} \cdot t} dt$ is $X_T [1/\overline{\widehat{X}} \cdot e^{\overline{\widehat{X}} \cdot t}]_T^\infty = -X_T/\overline{\widehat{X}} \cdot e^{\overline{\widehat{X}} \cdot T} > 0$ as $\overline{\widehat{X}} < 0$. Because $X_T < \infty$, the limit is finite. It follows that the integral $\int_T^\infty X_t dt$ converges.

We have thus proven that $\int_0^\infty X_t dt = \int_0^T X_t dt + \int_T^\infty X_t dt$ converges.

(ii) **Equality of solutions:** Because the integral $\int_0^\infty X_t dt$ converges, $\int_0^\infty X_t dt < F_0$ for a sufficiently large F_0. In this case, the natural resource constraint is not binding and it follows from (C.11) that $\lambda_{Rt} = 0$, $\forall t$. If $\lambda_R = 0$, differences in the first-order conditions compared to the baseline model only arise because labor is no longer used in intermediate production in the model of this section. But for parameter constellations such that there is quantity degrowth in the baseline model, labor use in intermediate production converges to zero in the baseline model as well, so that the first-order conditions and therefore the long-run solutions are identical for $t \to \infty$.