Pareto Optimality and Existence of Monetary Equilibria in a Stochastic OLG Model: A Recursive Approach

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Abstract

We provide a recursive approach for assessing Pareto efficiency in a pure endowment stochastic OLG model. Our approach is constructive and we derive from it a complete characterization of Pareto efficiency for stationary allocations on a continuous state space. We use our efficiency characterization to prove that a monetary equilibrium exists on a general state space if and only if the initial allocation is inefficient, extending existing results.

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1 Introduction

This paper provides a recursive approach for assessing Pareto efficiency in a pure endowment stochastic OLG model. We then make use of our recursive formulation to derive a complete characterization of Pareto efficiency for stationary allocations on a continuous state space. We use our efficiency characterization to prove that a monetary equilibrium exists if and only if the initial allocation is inefficient.

Our first contribution is to provide a recursive formulation for assessing Pareto inefficiency in stationary stochastic OLG economies with a general state space. The recursive formulation is based on a general-state-space extension of Chattopadhyay and Gottardi (1999)'s necessary condition for inefficiency in a pure exchange model with a discrete state space. Our suggestion is constructive and potentially computationally feasible. It is simpler to implement than an approach based directly on Chattopadhyay and Gottardi (1999)'s characterization. We reinterpret their approach for assessing Pareto efficiency, which is in the Cass tradition, as a minimax problem. We then apply methods from monotone dynamic programming along the lines of Bertsekas and Shreve (1978), chapter 5 [1]. Our approach suggest how to compute an *optimal* potentially improving transfer scheme as a stationary solution to a Bellman type of equation. Our method is not limited to the specific model considered here, but can be applied to any stochastic OLG model with a stationary structure (e.g. models with capital as an endogenous state variable as in Demange and Laroque (2000)).

Our second contribution is an application of the recursive formulation of the minimax problem. We restrict attention to stationary allocations on a general state space and characterize Pareto efficiency of stationary allocations.[2] This task resembles the dominant eigenvector characterization for models with a discrete state space. Our result provides an alternative characterization to the one of Manuelli (1990).[3] Also, our results extend the partial classification of efficiency in Demange and Laroque (2000). Bloise and Calciano (2008) obtain a characterization of robust inefficiency that has a form similar to our Theorem 2.

Our third contribution is another application of the recursive formulation, where we

[1] The results from chapter 5 in Bertsekas and Shreve (1978) are however not directly applicable, since some assumption made there (like Assumption I.2 on p. 71) do not hold in our minimax problem. We use the specific structure of our minimax problem to prove the results we need.

[2] Most papers on stochastic OLG models deal with a discrete shock space. Apart from the work cited subsequently, Zilcha (1990), who examines capital overaccumulation, is one of the few papers that deal with a continuous shock space.

[3] As we discuss subsequently and as has already been pointed out in Demange and Laroque (1999, 2000), Manuelli's characterization is not correctly proved and it is unclear to us whether it holds or not.

deal with the existence of monetary equilibria and the link between Pareto inefficiency and the existence of monetary equilibria in our stationary setup. We extend the results for a continuous state space from Manuelli (1990) and show that in a stationary set up there is the same close link between optimality of an (initial) allocation and the existence of (optimal) monetary equilibria, as is the case under certainty [see Benveniste and Cass (1986), Okuno and Zilcha (1983) and - for a setting with capital - Tirole (1985)] and under uncertainty with a discrete state space [see Aiyagari and Peled (1991)].[4]

Although we restrict attention to the relatively simple case of stationary endowment economies, our first and second contribution can be applied to more complicated economies with capital accumulation (as has been done in Barbie and Hillebrand (2015), where the topic of the third contribution of this paper is studied in such a framework), which naturally have a continuous state space. In our view it makes sense to develop the first two insights first in a relatively simple framework, which displays all essential elements of more involved set-ups, and then adapt them to these situations. So we view the first two contributions as the central methodological progress, and the third contribution as a first application of them.

The paper is organized as follows. Section 2 describes the pure endowment economy. In Section 3, a necessary condition for interim Pareto efficiency in a competitive equilibrium with continuous state space is derived. Section 4 provides a recursive approach to assess Pareto efficiency for stationary allocations. Section 5 presents a complete characterization of inefficiency in the stationary framework. Section 6 deals with the existence of monetary equilibria. Proofs are given in the appendix.

2 The Model

Uncertainty Time is discrete with $t = 0, 1, \ldots$. Uncertainty is described by a state space S, where S is a compact Polish space[5]. We denote by $\mathcal{S}$ the Borel $\sigma-$algebra on S. The stochastic evolution of the state $s \in S$ over time is described by a stochastic kernel $P(s, ds')$ from S to S with full support for each $s \in S$. Further, we assume that for all $s, s' \in S$ and $A \in \mathcal{S}$ we have $P(s, A) > 0$ if and only if $P(s', A) > 0$. Also, for fixed $A \in \mathcal{S}$, the mapping $P(s, A)$ from S to the real numbers is $\mathcal{S}-$measurable. Let s^t denote

[4]Other related work includes Balasko and Shell (1981) and Okuno and Zilcha (1980) who deal with optimal monetary equilibra in a deterministic non-stationary framework as well as Gottardi (1996) who examined the existence of monetary equilibria in a stationary stochastic OLG model with incomplete markets.

[5]As any finite set is a compact Polish space under the topology generated by the discrete metric, the finite state space case is trivially included.

the time t history $s^t = (s_0, s_1, ..., s_t)$, $s_i \in S$. For each history s^t, we denote by s_i^t for $i = 0, 1, ..., t$ the i-th coordinate of s^t.

Define $S^\infty := \prod_{t=0}^{\infty} S_t$ and denote by $\mathcal{S}^\infty$ the product Borel $\sigma-$algebra on S^∞, where $S_t = S$ for each $t = 0, 1....$ For each $s^\infty \in S^\infty$, we denote by s_i^∞ the i-th coordinate of s^∞ for $i = 0, 1, ...$ Let $S^t := \prod_{i=0}^{t} S_t$ and denote by $\mathcal{S}^t$ the Borel $\sigma-$algebra on S^t. By the Kolmogorov extension theorem, the transition probabilities $P(s, .)$ together with an initial probability measure μ on $(S, \mathcal{S})$ generate a unique consistent probability measure P^∞ on $(S^\infty, \mathcal{S}^\infty)$. We fix some arbitrary μ throughout the paper.

Aggregate Endowment The aggregate endowment in each history is described by a function $\omega(s)$ such that ω is $\mathcal{S}-$measurable, strictly positive and bounded, i.e. for some finite K we have $K > \omega(s) > 0$ for all $s \in S$.[6]

Consumers Each period t, a new consumer is born who lives for two periods. We assume that the consumers born in period t are distinguished by the history up to time t. Thus each consumer faces risk in his second period of life (old age), but not in his first period (young age). There is one consumption good in each history s^t. Each consumer has consumption for his young age and old age, depending on the state of nature. There is one initially old consumer in period 0.

Allocations A general (non-stationary) allocation is defined by a family of functions $\{(c_t^y(s^\infty), c_t^o(s^\infty))\}_{t=0}^{\infty}$ such that young age consumption c_t^y and old age consumption c_t^o are $\mathcal{S}^t-$measurable[7] for each t and $c_t^y(s^\infty) + c_t^o(s^\infty) = \omega(s_t)$ for each $s^\infty \in S^\infty$ and each t. An allocation is called *interior* if there exists a $k > 0$ such that $c_t^y(s^\infty) > k$ and $c_t^o(s^\infty) > k$ for each $s^\infty \in S^\infty$ and each t.

We mainly focus on stationary consumption allocations. For any given s we denote by $c^y(s)$ the stationary allocation of young age consumption and by $c^o(s)$ the stationary allocation of old age consumption. Both are $\mathcal{S}-$measurable function and thus depend only on the current state s, but not on the whole history s^t. We denote stationary allocations by $\{(c^y(s), c^o(s))\}_{s\in S}$.

Preferences Preferences for a consumer born in history s^t with $t \geq 0$ are given by $U(c_t^y, c_{t+1}^o, s^t) := \int_S u(c_t^y(s^t), c_{t+1}^o(s^t, s')) P(s_t, ds')$ for an allocation $\{(c_t^y(s^\infty), c_t^o(s^\infty))\}_{t=0}^{\infty}$. u is twice continuously differentiable, strictly increasing in both arguments with negative definite Hessian matrix. The preferences of the initially old consumers in period 0 who live in state s_0 are strictly increasing in consumption $c^o(s_0)$. Define $U(c^y, c^o, s_t) := \int_S u(c^y(s_t), c^o(s')) P(s_t, ds')$ for a stationary allocation $\{(c^y(s), c^o(s))\}_{s\in S}$. With a slight

[6] We say a function f on S is strictly positive if $f(s) > 0$ for each $s \in S$.

[7] Since the function c_t^y is $\mathcal{S}^t-$measurable, we use the notation $c_t^y(s^t)$ as well as $c_t^y(s^\infty)$. The same applies for other $\mathcal{S}^t-$measurable functions.

abuse of notation we set $U(.,.,.) = c_0^o(s_0)$ for the initial old.

Supporting Prices We define $m(s,s') := \frac{u_2(c^y(s),c^o(s'))}{\int_S u_1(c^y(s),c^o(s'))P(s,ds')}$ for a stationary allocation. Our assumptions on the transition probabilities ensure that $m(s,s')$ is a $\mathcal{S}\otimes\mathcal{S}-$measurable function.[8] The functions m serve as supporting prices for a given stationary allocation.[9] They will play a key role in determining whether a given stationary allocation is efficient.

3 A Necessary Condition for Inefficiency

We first define the notion of efficiency we adopt [see Muench (1977) and Peled (1982)].

Definition 1 *A stationary allocation* $(c^y(s), c^o(s'))$ *is interim Pareto efficient if there does not exist an allocation* $\{(c_t^y(s^\infty), c_t^o(s^\infty))\}_{t=0}^\infty$ *such that*

$$U(c^y, c^o, s_t) \leq U\left(c_t^y, c_{t+1}^o, s^t\right)$$

for each t *and* $s^\infty \in S^\infty$*, with strict inequality on a set* $\widetilde{A}$ *of strictly positive* P^∞*-measure, i.e. for any* $s^\infty \in \widetilde{A}$ *there exists some* t *with* $U(c^y, c^o, s_t) < U\left(c_t^y, c_{t+1}^o, s^t\right)$.

For each s^t, define the set of measurable *weight functions* [Chattopadhyay and Gottardi (1999)] as $\mathcal{U}(s^t) = \left\{\lambda(s^t,s') \geq 0 : \ \lambda(s^t,.) \text{ is } \mathcal{S}-\text{measurable with } \int_S \lambda(s^t,s')P(s_t,ds') = 1\right\}$.[10] For any path $s^\infty \in S^\infty$, we define $(s^\infty)^t$ as the history up to time t along this path, i.e. $(s^\infty)^t = (s_0^\infty, s_1^\infty, ..., s_t^\infty)$ for any $t \geq 0$.

As a preliminary step towards our recursive formulation we generalize the necessary condition of Chattopadhyay and Gottardi (1999) for Pareto inefficiency from the case of a discrete shock space to our more general setup:

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[8] To see this, define a new transition probability by $\widetilde{P}(s, A\times B) = \int_B P(\widetilde{s}, A)\,d\delta_s(\widetilde{s})$ for any $A, B \in \mathcal{S}$ and apply Theorem 18.7 in Aliprantis and Border (1999) to the $\mathcal{S}\otimes\mathcal{S}-$measurable function $u_1(c^y(s), c^o(s'))$ and $\widetilde{P}$. δ_s denotes the Dirac measure in $s \in S$.

[9] Since there is only one person (or a group of identical persons) born in each history s^t, any allocation will be short-run efficient in the sense that it is not possible to interim Pareto improve by a reallocation that only changes the allocation of persons born up to some time t. Therefore we do not have to assume that the allocations we consider are competitive equilibrium allocations, as is necessary with heterogenous agents who are living during the same periods [see e.g Chattopadhyay and Gottardi (1999)].

[10] The definition of a weight function $\widetilde{\lambda}$ given in Chattopadhyay and Gottardi (1999) is slightly different from ours. Because Chattopadhyay and Gottardi (1999) use an abstract date-event tree setting without objective probabilities, their definition in our expected utility set-up for the special case where S is finite amounts to setting $\widetilde{\lambda}(s^t,s') = P(s_t,s')\cdot\lambda(s^t,s')$. With their definition, $\sum_{s'\in S}\widetilde{\lambda}(s^t,s') = 1$ and the concepts are equivalent (using our condition that $\sum_{s'\in S}\lambda(s^t,s')\cdot P(s_t,s') = 1$). The condition for inefficiency remains unaffected by the definition of the weight function since our supporting prices $m(s,s')$ also do not contain the transition probabilities.

Proposition 1 *If an interior stationary allocation is not interim Pareto efficient, there exists a set $A \in \mathcal{S}^t$ of strictly positive P^∞-measure, a $\mathcal{S}^t-$measurable function C defined on A and a family of functions $\lambda\left(s^{t+i}, s'\right) \in \mathcal{U}\left(s^{t+i}\right), i = 0, 1, ...$ such that*

$$\sum_{i=0}^{\infty} \prod_{j=0}^{i} \frac{\lambda\left(\left(s^\infty\right)^{t+j}, s^\infty_{t+j+1}\right)}{m\left(s^\infty_{t+j}, s^\infty_{t+j+1}\right)} \leq C\left(s^t\right) \tag{1}$$

for each path $s^\infty \in A$.

Remark 1 *It is a straightforward although somewhat lengthy extension of Chattopadhyay and Gottardi (1999) to prove that that condition* (1) *is sufficient for a stationary allocation to be interim Pareto suboptimal in our setup.*

Remark 2 *It should be noted that in general we cannot expect to obtain a constant C (independent of s^t) on the right-hand side of equation (1). This follows from the proof of Proposition 1 and is consistent with Chattopadhyay and Gottardi (1999), who only consider a single s^t, not a set $A \in \mathcal{S}^t$.*

4 Recursive Formulation

The necessary condition for interim Pareto optimality for OLG economies given in Proposition 1 uses the full set of supporting prices m to give an answer about the efficiency of a stationary allocation. But even if one knows the prices of all state contingent claims, one still has to test for all weight functions λ and check whether one obtains convergence or divergence in (1). This makes it difficult to decide on interim Pareto efficiency by applying condition (1) directly. In this section, we propose a recursive formulation of the condition for interim Pareto inefficiency with the space of shocks S as the state space.

The idea is to reinterpret the condition of Chattopadhyay and Gottardi (1999) (which we have generalized to our setup) as a minimax problem. First, for a fixed weight function λ the sum over the product of the (relative) contingent claims prices for the path that attains the highest sum is computed. Second, the weight function is computed that minimizes the outcome on these "highest-sum" paths. If the procedure leads to a finite value of the sum, then Pareto inefficiency is detected. This procedure can be expressed recursively, as we will show.

Define $\mathcal{U}^\infty\left(s_0\right) := \prod_{s^t} \mathcal{U}\left(s^t\right)$, where the product is taken over all histories s^t with $s^t_0 = s_0$ for some $s_0 \in S$. Define $S^\infty_0 = \{s_0\} \times \prod_{i=1}^\infty S_i$. For any given starting state $s_0 \in S$

we consider

$$\inf_{\lambda^{\infty}\in\mathcal{U}^{\infty}(s_0)} \sup_{s^{\infty}\in S_0^{\infty}} \quad 1+\sum_{i=0}^{\infty}\prod_{j=0}^{i}\frac{\lambda\left(\left(s^{\infty}\right)^{j}, s_{j+1}^{\infty}\right)}{m\left(s_j^{\infty}, s_{j+1}^{\infty}\right)} \tag{2}$$

We denote by $J^*(s_0)$ the value of this minimax problem. Note that because of the stationarity of the allocation, the value J^* for an economy starting after some history s^t will only depend on the state s_t at which the economy starts. Thus the set $A \in \mathcal{S}^t$ in Proposition 1 takes the form $A = S \times ... \times S \times \widetilde{A}$ for some $\widetilde{A} \in \mathcal{S}$. Also, $P^{\infty}(A) > 0$ only if $P\left(s, \widetilde{A}\right) > 0$ for some $s \in S$. From our assumption on the transition probabilities we then have $P\left(s', \widetilde{A}\right) > 0$ for any $s' \in S$ and say that $\widetilde{A}$ *has positive measure.* Thus:

Fact *If a stationary allocation is Pareto inefficient, there must exist a set of strictly positive measure $\widetilde{A} \in \mathcal{S}$ such that for each $s \in \widetilde{A}$ we have $J^*(s) < \infty$.*

In fact, we will subsequently in the proof of Proposition 3 show that $J^*(s) < \infty$ for all $s \in S$ in this case. It is the goal of this section to show that the problem (2) can be written in a recursive way and that the value function J^* can be computed as a pointwise limit by successively applying the recursive formulation to some starting function J_0.

First, for each $s \in S$ we denote the set of all stationary weight functions by $\mathcal{U}(s) = \left\{\lambda(s, s') \geq 0,\ \lambda(s, .) \text{ is } \mathcal{S}\text{−measurable with } \int_S \lambda(s, s') P(s, ds') = 1\right\}$. Define the Bellman equation:

$$T(J)(s) := 1 + \inf_{\lambda(s)\in\mathcal{U}(s)} \sup_{s'\in S} \frac{\lambda(s, s')}{m(s, s')} \cdot J(s') \tag{3}$$

for any nonnegative extended real valued function J. The Bellman operator T allows to obtain the desired recursive formulation of (2):

Theorem 1 *The value function J^* can be obtained as a fixed point to the Bellman equation (3), i.e. $J^* = T(J^*)$.*

Now we use the Bellman operator T to compute *stepwise* a value function that solves the Bellman equation (3). T^N denotes applying N times the operator T. For each $s \in S$ we can as in Bertsekas and Shreve (1978) define $J_{\infty}(s) := \lim_{N\to\infty} T^N(J_0)(s)$, where we set $J_0 \equiv 1$. Note that the limit exists since the sequence $T^N(J_0)(s)$ is by definition increasing. We now show that:

Proposition 2 (a) *J_{∞} can be used to compute the value function J^* as a monotone limit:*

$$J_{\infty} = T(J_{\infty}) = T(J^*) = J^*. \tag{4}$$

(b) *J^* is a $\mathcal{S}-$measurable function.*

It may be of interest that Theorem 1 allows to show that in problem (2) the infimum is achieved for some λ^∞ and the weights doing this have a markovian structure. More precisely, we have

Proposition 3 *There exist $\lambda^*(s) \in \mathcal{U}(s)$ for each $s \in S$ such that*

$$J^*(s_0) = \sup_{s^\infty \in S_0^\infty} 1 + \sum_{i=0}^{\infty} \prod_{j=0}^{i} \frac{\lambda^*\left(s_j^\infty, s_{j+1}^\infty\right)}{m\left(s_j^\infty, s_{j+1}^\infty\right)} \tag{5}$$

for each $s_0 \in S$.

5 Complete Characterization of Efficiency of Stationary Allocations

This section is a first application of the recursive formulation of the minimax problem related to the Chattopadhyay and Gottardi (1999) characterization of interim Pareto inefficiency in OLG models. We restrict attention to stationary allocations in this section and characterize interim Pareto efficiency of a stationary allocation. Our main result in this section, Theorem 2, is closely related to a number of contributions to the literature on efficiency of stationary allocations in OLG models, namely Manuelli (1990), Demange and Laroque (1999, 2000) as well as Aiyagari and Peled (1991).

Aiyagari and Peled (1991) gave a dominant root characterization of Pareto optimality of stationary allocations within the class of stationary allocations on a discrete shock space. This was extended by Demange and Laroque (1999) and Chattopadhyhay and Gottardi (1999, Theorem 4) as a *general* characterization of interim Pareto optimality of stationary allocations, i.e. a characterization of Pareto optimality of stationary allocations within the class of *all* allocations.

Manuelli (1990, Theorem 3) derived a characterization of interim Pareto optimality in a model with an infinite shock space that says an allocation is interim Pareto optimal[11] if and only if the reverse inequality of (7) with $\leq$ instead of $>$ holds for a bounded measurable function η. Clearly, our Theorem 2 (b) and his Theorem 3 are not necessarily equivalent as long as there is no such thing as a dominant root characterization of optimality for

[11] As already noted in Demange and Laroque (1999, 2000), it is, however, sometimes difficult to follow Manuelli's proof. In particular, the proof of the necessity of the inequality condition for Pareto optimality is totally unclear and we are not sure whether a result going beyond Demange and Laroque (2000) holds.

the case of an infinite state space. Apart from this earlier literature Bloise and Calciano (2008) characterize *robust inefficiency* in their Proposition 2 with a condition that has some similarity with our condition.

In a somewhat different set-up with production and capital as a state variable, but with a finite number of exogenous shocks, Demange and Laroque (2000), show in their Theorem 1 that (7) is sufficient for interim Pareto inefficiency, and in their Theorem 2 that the reverse inequality of (7) with $\leq$ instead of $>$ is sufficient for interim Pareto optimality. Our Lemma 4 in the appendix shows the same result as Theorem 2 in Demange and Laroque (2000). They did, however, not derive a complete characterization of optimality (see also their Theorem 3 for a partial classification in terms of the spectral radius of a positive operator, the analogy of the dominant root of a positive matrix).

Proposition 4 *If a stationary allocation is interim Pareto inefficient, there exists a strictly positive measurable function* $\eta(s)$ *that is bounded above such that*

$$\int_S m(s,s') \cdot \eta(s') P(s,ds') > \eta(s) \quad \textit{for all } s \in S. \tag{6}$$

The function η *can be computed as the monotone limit (or as a fixed point of* T*):* $\eta(s) = \lim_{N\to\infty} \frac{1}{T^N(J_0)(s)} = \frac{1}{J_\infty(s)} = \frac{1}{J^*(s)}$ *for all* $s \in S$.

To show the converse implication, namely that the existence of such a function η implies interim Pareto inefficiency we need to add the following continuity assumptions on the transition probabilities:

Assumption 1 *The transition probability* P *has the Feller property, i.e.* $\int_S f(s') P(s,ds')$ *is continuous in* s *for each (bounded) continuous function* f.

Assumption 2 *If* $s_n \to s$*, then* $P(s_n, A) \to P(s, A)$ *for each* $A \in \mathcal{S}$.

Assumption 2 implies Assumption 1 (see Stokey and Lucas with Prescott (1989)). Assumption 1 is fullfilled in the set-up of Demange and Laroque (2000) under their assumption of a continuous Markov policy (see p.6 in Demange and Laroque (2000)) in which the state space S consists of exogenous (finite) shocks and the endogenous variable capital stock.

We restrict attention to continuous stationary allocation, i.e. stationary allocations where $c^y(s)$ and $c^o(s')$ are positive continuous functions on S. Note that since S is compact, such an allocation will automatically be interior. Under these continuity assumptions on P we can also strengthen the claim in the previous proposition. We show:

Theorem 2 **(a)** *Let Assumption 1 hold. If a stationary continuous allocation is interim Pareto inefficient, a upper semicontinuous function* η *such that* (6) *holds exists. Conversely, if for any continuous stationary allocation there exists a strictly positive continuous function* η *such that* (6) *holds, this allocation will be interim Pareto inefficient.*

(b) *Under Assumption 2, a continuous stationary allocation is interim Pareto inefficient if and only if there exists a strictly positive continuous function* η *such that*

$$\int_S m(s,s') \cdot \eta(s')\, P(s,ds') > \eta(s) \quad \text{for all } s \in S. \tag{7}$$

The function η *can be computed as the monotone limit (or as a fixed point of* T*):* $\eta(s) = \lim_{N\to\infty} \frac{1}{T^N(J_0)(s)} = \frac{1}{J_\infty(s)} = \frac{1}{J_*(s)}$ *for all* $s \in S$.

Remark 3 *The proof of the only-if part reveals that ineffciency is indeed implied if there exists a strictly positive bounded measurable function* η *on* S *and a* $\delta > 0$ *such that* $\int_S m(s,s')\, \eta(s')\, P(s,ds') - \eta(s) > \delta > 0$ *for all* $s \in S$.[12]

It is of some interest to consider the special case in which utility is additively separable over time and shocks are i.i.d.. In this case, utility can be represented as $u(c^y(s)) + \int_S v(c^o(s'))\, P(ds')$, where P is the probability measure according to which the shocks are distributed and u, v are twice continuously differentiable, strictly increasing and concave with negative second derivatives. Of course, the preferences of the initially old generation remains unaffected.
In this case, we do not have to compute the function η explicitly. We have:

Corollary 1 *Under additive separable utility and i.i.d. shocks, a continuous stationary allocation is interim Pareto inefficient if and only if*

$$\int_S \frac{v'(c^o(s'))}{u'(c^y(s'))} P(ds') > 1. \tag{8}$$

6 Monetary Equilibria

In this section, the utility function is assumed to be additively separable and takes the form $u(c^y(s)) + \int_S v(c^o(s'))\, P(s,ds')$. Otherwise it satisfies all the assumptions from the previous sections, and in addition we assume for u the Inada condition $\lim_{x\to 0} u'(x) = \infty$.

[12]We would like to thank an anonymous referee for pointing this out.

Furthermore, we impose a stronger continuity requirement on the transition probabilities than in the previous section. Let $\mathcal{P}(S)$ be the set of probability measure on S. For $\lambda \in \mathcal{P}(S)$, the total variation norm $\|\lambda\|_{TV}$ is the norm defined by $\|\lambda\|_{TV} := \sup \sum_{i=1}^{k} |\lambda(A_i)|$, where the supremum is taken over all finite partitions of S into disjoint measurable subsets. We have that $\|\lambda_n - \lambda\|_{TV} \to 0$ if and only if $\lim_{n\to\infty} |\lambda_n(A) - \lambda(A)| = 0$ for all $A \in \mathcal{S}$ and the convergence is uniform in A. Thus convergence in the variation norm requires setwise convergence to hold uniformly (over all measurable subsets). This strengthens our continuity requirement from Assumption 2 for the transition probabilities from the previous section [see Stokey and Lucas with Prescott (1989), chapter 11 for a more detailed account].

Assumption 3 *The mapping $P : S \to \mathcal{P}(S)$ given by $P(s, \cdot)$ is continuous, where $\mathcal{P}(S)$ is endowed with the total variation norm.*

In comparison to the continuity requirement from p. 271 in Manuelli (1990), Manuelli's condition requires Lipschitz continuity for the transition probability with respect to a special integrand, which is determined from the utility function. We could also impose such a condition, but we think it is more natural to have a separate condition on the transition probabilities.

To motivate our definition of monetary equilibrium, consider an intrinsically useless asset that is owned by the initial old. Each initial old person is endowed with unit of this asset and his old age consumption. Apart from this, all persons in the economy are endowed with a stationary allocation $\{c^y(s), c^o(s)\}_{s\in S}$. A monetary equilibrium is given if the market for the asset called *money* clears after each history at a strictly positive price for the asset. From the first order conditions of the consumers, this gives the following definition:

Definition 2 *A stationary monetary equilibrium is given by a continuous stationary allocation $\{c^y(s), c^o(s)\}_{s\in S}$ and a continuous strictly positive function $p(s)$ with $p(s) \le c^y(s)$ for each $s \in S$ such that the following first order condition with respect to an asset with price p holds:*

$$\int_S \frac{v'(c^o(s') + p(s'))}{u'(c^y(s) - p(s))} \cdot p(s')\, P(s, ds') = p(s) \quad \textit{for all } s \in S.$$

We say that such a tuple $\{c^y(s), c^o(s), p(s)\}_{s\in S}$ is a *monetary equilibrium associated with allocation* $\{c^y(s), c^o(s)\}_{s\in S}$. This definition is equivalent to the one of Manuelli

(1990).[13] The characterization result of this section is:

Theorem 3 *Let a continuous stationary allocation be given and let Assumption 3 hold. Then an allocation is interim Pareto inefficient if and only if there exists a monetary equilibrium associated with it. Furthermore, the monetary equilibrium allocation* $\{c^y(s) - p(s), c^o(s) + p(s)\}_{s\in S}$ *is interim Pareto efficient.*

Our results contain the results of Manuelli (1990) in Theorem 1 and 2 of his paper as a special case. In his Theorem 1, Manuelli gives a sufficient condition for the existence of a monetary equilibrium. He shows that if the condition

$$\min_{s\in S} \int_S \frac{v'(c^o(s'))}{u'(c^y(s'))} P(s, ds') > 1$$

holds, then a monetary equilibrium exists. His condition holds if and only if our inefficiency condition (7) holds for the special choice $\eta(s) = \frac{1}{u'(c^y(s'))}$. His Theorem 1 is therefore an immediate consequence of our Theorems 2 and 3. In his proof, Manuelli (1990) also construct of fixed point mapping as we do in Lemma 1, however mapping from the left-hand to the right-hand side of the Euler equation. He then uses a different fixed point argument due to his different assumptions on transition probabilities etc.. We use instead a vanishing dividends argument that goes back in its basic form, for a finite state context, where instead of Euler equations a classical Arrow-Debreu existence proof is used, to Aiyagari and Peled (1991).

Manuelli's Theorem 2 is a necessary and sufficient condition for the existence of a monetary equilibrium in the case of separable utility and i.i.d. shocks. He shows that a monetary equilibrium exists if and only if (8) holds. With our Corollary 1 and our Theorem 3 this also follows immediately from our results.

Appendix: Proofs

Proposition 1

Proof. Let $c^y(s)$ and $c^o(s')$ be the given stationary inefficient allocation and let $\{(\widetilde{c}^y_t(s^\infty), \widetilde{c}^o_t(s^\infty))\}_{t=0}^\infty$ be an improving allocation. For each history s^t define $\varepsilon^y_t(s^t) = \widetilde{c}^y_t(s^t) - c^y(s_t)$ and $\varepsilon^o_t(s^t) = \widetilde{c}^o_t(s^t) - c^o(s_t)$. For each path s^∞, define $T(s^\infty) = \min\{t \in \mathbb{N}\cup\{\infty\} \,|\varepsilon^o(s^t) \neq 0\}$. It is easy to see that T is a stopping time and hence that $\{s^\infty \,|T(s^\infty) = t\} \in \mathcal{S}^t$ for each t. Since $P^\infty\left(\widetilde{A}\right) > 0$, given the definition of $\widetilde{A}$,

[13] See Manuelli (1990), p.273 equation (1). Manuelli does not have a constant stock of money (equal to 1) as we have, but a stock of money that changes over time. This difference is, however, inessential for the results and our assumption on the stock of money follows by setting his function $g(s, s') = 1$.

we must have $P^{\infty}(\{T<\infty\})>0$. Since $\{T<\infty\}=\cup_{t=0}^{\infty}\{T=t\}$, there must exist some t with $P^{\infty}(\{T=t\})>0$. Since the new allocation is interim Pareto improving, we must have that $P^{\infty}(\{T=t\}\cap\{s^{\infty}\,|\varepsilon_t^o(s^t)>0\})>0$. Take $A:=\{T=t\}\cap\{s^{\infty}\,|\varepsilon_t^o(s^t)>0\}$.

Since the original stationary allocation is interior, we have $k<c^y(s)<K$ and $k<c^o(s')<K$ for all $s,s'\in S$. Since the utility functions are concave, we can assume that $0<k-\epsilon<\widetilde{c}_t^y(s^{\infty}),\widetilde{c}_t^o(s^{\infty})<K$ for $\epsilon>0$ for the improving allocation. Now we have for each path s^{∞} by a second order Taylor expansion (where $\langle,\rangle$ denotes the inner product in euclidian space):

$$\begin{aligned}
& u\left(\widetilde{c}_t^y(s^{\infty}),\widetilde{c}_{t+1}^o(s^{\infty})\right)-u\left(c^y(s_t^{\infty}),c^o(s_{t+1}^{\infty})\right)\\
= & \left\langle Du\left(c^y(s_t^{\infty}),c^o(s_{t+1}^{\infty})\right),\left(\varepsilon_t^y\left((s^{\infty})^t\right),\varepsilon^o\left((s^{\infty})^{t+1}\right)\right)\right\rangle\\
& +\frac{1}{2}\left(\varepsilon_t^y\left((s^{\infty})^t\right),\varepsilon^o\left((s^{\infty})^{t+1}\right)\right)'D^2u\left(\xi_t^y\left((s^{\infty})^t\right),\xi_{t+1}^o\left((s^{\infty})^{t+1}\right)\right)\left(\varepsilon_t^y\left((s^{\infty})^t\right),\varepsilon_{t+1}^o\left((s^{\infty})^{t+1}\right)\right),
\end{aligned}$$

where $\xi_t^y(s^t)$ and $\xi_{t+1}^o(s^{t+1})$ are from the compact set $[k-\epsilon,K]$.

Since matrix $D^2u\left(\xi_t^y(s^t),\xi_{t+1}^o(s^{t+1})\right)$ is negative definite, and the function u is twice continuously differentiable, we have ($\|.\|$ denotes the euclidian norm):

$$\begin{aligned}
& -\frac{1}{2}\left(\varepsilon_t^y\left((s^{\infty})^t\right),\varepsilon^o\left((s^{\infty})^{t+1}\right)\right)'D^2u\left(\xi_t^y\left((s^{\infty})^t\right),\xi_{t+1}^o\left((s^{\infty})^{t+1}\right)\right)\left(\varepsilon_t^y\left((s^{\infty})^t\right),\varepsilon_{t+1}^o\left((s^{\infty})^{t+1}\right)\right)\\
\geq & H\left(\xi_t^y\left((s^{\infty})^t\right),\xi_{t+1}^o\left((s^{\infty})^{t+1}\right)\right)\cdot\left\|\varepsilon_t^y\left((s^{\infty})^t\right),\varepsilon_{t+1}^o\left((s^{\infty})^{t+1}\right)\right\|^2
\end{aligned}$$

for some strictly positive continuous function H.[14] Since ξ are elements of a compact set for each history s^t, there is a strictly positive constant H such that for all histories $(s^{\infty})^t$:

$$\begin{aligned}
& -\frac{1}{2}\left(\varepsilon_t^y\left((s^{\infty})^t\right),\varepsilon^o\left((s^{\infty})^{t+1}\right)\right)'D^2u\left(\xi_t^y\left((s^{\infty})^t\right),\xi_{t+1}^o\left((s^{\infty})^{t+1}\right)\right)\left(\varepsilon_t^y\left((s^{\infty})^t\right),\varepsilon^o\left((s^{\infty})^{t+1}\right)\right)\\
\geq & H\cdot\left\|\varepsilon_t^y\left((s^{\infty})^t\right),\varepsilon^o\left((s^{\infty})^{t+1}\right)\right\|^2.
\end{aligned}$$

Thus we have for each given history s^t

$$\begin{aligned}
0 \leq & \int_S u\left(\widetilde{c}_t^y(s^t),\widetilde{c}_{t+1}^o(s^t,s')\right)P(s_t,ds')-\int_S u\left(c^y(s_t),c^o(s')\right)P(s_t,ds')\\
\leq & \int_S\left\langle Du\left(c^y(s_t),c^o(s')\right),\left(\varepsilon_t^y(s^t),\varepsilon^o(s^t,s')\right)\right\rangle P(s_t,ds')\\
& -\frac{1}{2}H\int_S\left\|\varepsilon^y(s^t),\varepsilon^o(s^t,s')\right\|^2P(s_t,ds').
\end{aligned}$$

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[14]For any positive definite matrix A, the problem

$$\min_{\{x\,|\|x\|\leq 1\}}\frac{x'}{\|x\|}A\frac{x}{\|x\|}$$

has a strictly positive solution ϑ, which by the Berge maximum theorem is a continuous function of the entries of the matrix A.

It follows that

$$\int_S u_2(s_t,s')\cdot\varepsilon^o\left(s^t,s'\right)P(s_t,ds')\geq-\varepsilon^y\left(s^t\right)\int_S u_1(s_t,s')P(s_t,ds')+\frac{1}{2}H\left(\varepsilon^y\left(s^t\right)\right)^2.$$

This is equivalent to

$$\int_S\frac{u_2(s_t,s')}{\int_S u_1(s_t,s')P(s_t,ds')}\cdot\varepsilon^o\left(s^t,s'\right)P(s_t,ds')\geq-\varepsilon^y\left(s^t\right)+\frac{1}{2}\frac{H\left(\varepsilon^y\left(s^t\right)\right)^2}{\int_S u_1(s_t,s')P(s_t,ds')}.$$

Since $u_1(s_t,s')$ is bounded above, by continuity and compactness, there is some $\rho>0$ such that (using that $\varepsilon^o\left(s^t\right)=-\varepsilon^y\left(s^t\right)$):

$$\int_S\frac{u_2(s_t,s')}{\int_S u_1(s_t,s')P(s_t,ds')}\cdot\varepsilon^o\left(s^t,s'\right)P(s_t,ds')\geq\varepsilon^o\left(s^t\right)+\rho\left(\varepsilon^o\left(s^t\right)\right)^2$$

If we define $\varepsilon^o_+\left(s^t\right):=\max\left\{\varepsilon^o\left(s^t\right),0\right\}$ for each history s^t, we have

$$\int_S\frac{u_2(s_t,s')}{\int_S u_1(s_t,s')P(s_t,ds')}\cdot\varepsilon^o_+\left(s^t,s'\right)P(s_t,ds')\geq\varepsilon^o_+\left(s^t\right)+\rho\left(\varepsilon^o_+\left(s^t\right)\right)^2. \tag{9}$$

For each $s^t\in A$, we define now $\lambda\left(s^t,s'\right):=\frac{\frac{u_2(s_t,s')}{\int_S u_1(s_t,s')P(s_t,ds')}\cdot\varepsilon^o_+\left(s^t,s'\right)}{\int_S\frac{u_2(s_t,s')}{\int_S u_1(s_t,s')P(s_t,ds')}\cdot\varepsilon^o_+(s^t,s')P(s_t,ds')}$. Note that $\int_S\frac{u_2(s_t,s')}{\int_S u_1(s_t,s')P(s_t,ds')}\cdot\varepsilon^o_+\left(s^t,s'\right)P(s_t,ds')>0$ since $\varepsilon^o_+\left(s^t\right)>0$. Taking the inverse in (9), we obtain

$$\frac{1}{\int_S\frac{u_2(s_t,s')}{\int_S u_1(s_t,s')P(s_t,ds')}\cdot\varepsilon^o_+\left(s^t,s'\right)P(s_t,ds')}\leq\frac{1}{\varepsilon^o_+\left(s^t\right)+\rho\left(\varepsilon^o_+\left(s^t\right)\right)^2}.$$

Using the definition of λ, for states s' with $\varepsilon^o_+\left(s^t,s'\right)>0$ this can be written as

$$\lambda\left(s^t,s'\right)\cdot\frac{1}{\frac{u_2(s_t,s')}{\int_S u_1(s_t,s')P(s_t,ds')}\cdot\varepsilon^o_+\left(s^t,s'\right)}\leq\frac{1}{\varepsilon^o_+\left(s^t\right)+\rho\left(\varepsilon^o_+\left(s^t\right)\right)^2}.$$

Continuing as in Chattopadhyay and Gottardi (1999), starting from some history $s^t\in A$, we obtain for any path with $\varepsilon^o_+\left(s^u,s'\right)>0$ for all s^u, $u\geq t$,

$$\prod_{j=0}^{u-t}\frac{\lambda\left(\left(s^\infty\right)^{t+j},s^\infty_{t+j+1}\right)}{m\left(s^\infty_{t+j},s^\infty_{t+j+1}\right)}\cdot\frac{1}{V\cdot\varepsilon^o_+\left(s^{u+1}\right)}+\widetilde{\rho}\sum_{i=0}^{u-t-1}\prod_{j=0}^{i}\frac{\lambda\left(\left(s^\infty\right)^{t+j},s^\infty_{t+j+1}\right)}{m\left(s^\infty_{t+j},s^\infty_{t+j+1}\right)}\leq\frac{1}{V\cdot\varepsilon^o_+\left(s^t\right)}-\widetilde{\rho}$$

where V and $\widetilde{\rho}$ are some positive constants. Thus, for $C\left(s^t\right):=\frac{1}{V\varepsilon^o_+\left(s^t\right)}-\widetilde{\rho}$ we obtain the formula of the proposition. For all other paths, we have $\lambda\left(s^u,s'\right)=0$ after some history and the same formula holds. This proves the proposition. ■

Theorem 1

Proof. We first show $J^*(s) \leq T(J^*)(s)$ for all $s \in S$. Note that for any $\lambda(s) \in \mathcal{U}(s)$ from the definition of $J^*(s')$ as the infimum, we can choose "continuation λ's", i.e. $\lambda^\infty \in \mathcal{U}^\infty(s')$ for the minimax problem starting with $s' \in S$, such that

$$J^*(s) \leq 1 + \sup_{s' \in S} \frac{\lambda(s,s')}{m(s,s')} \cdot (J^*(s') + \varepsilon(s,s')) \tag{10}$$

where $\varepsilon(s,s') = \frac{m(s,s')}{\lambda(s,s')} \cdot \varepsilon$ for some $\varepsilon > 0$ if $\lambda(s,s') > 0$ and ε if $\lambda(s,s') = 0$. To see this, note that any $\lambda^\infty \in \mathcal{U}^\infty(s)$ can be written as $(\lambda_1, (\lambda^\infty_{-1}(s'))_{s' \in S})$, where $\lambda_1 \in \mathcal{U}(s)$ and $\lambda^\infty_{-1}(s') \in \mathcal{U}^\infty(s')$ for each $s' \in S$. Further, note that

$$J^*(s) \leq 1 + \sup_{s^\infty \in S^\infty_s} \left(\frac{\lambda_1(s, s^\infty_1)}{m(s, s^\infty_1)} \left(1 + \sum_{i=1}^{\infty} \prod_{j=1}^{i} \frac{\lambda_{-1}((s^\infty)^j, s^\infty_{j+1})}{m(s^\infty_j, s^\infty_{j+1})} \right) \right)$$

where $S^\infty_s := \{s\} \times S \times S \times ....$ Denote the right-hand side by $M(\lambda_1, (\lambda^\infty_{-1}(s'))_{s' \in S}))$. Choose $\lambda^\infty_{-1}(s^\infty_1)$ for each $s^\infty_1 \in S$ such that

$$\left(1 + \sum_{i=1}^{\infty} \prod_{j=1}^{i} \frac{\lambda_{-1}((s^\infty)^j, s^\infty_{j+1})}{m(s^\infty_j, s^\infty_{j+1})} \right) \leq J^*(s^\infty_1) + \varepsilon(s, s^\infty_1) \tag{11}$$

for all $s^\infty \in \{s_0\} \times \{s^\infty_1\} \times S \times S \times ....$ (clearly such a choice is always possible). Let $s^\infty_n \in S^\infty_s$ be a sequence such that

$$1 + \left(\frac{\lambda_1(s, s^\infty_{1n})}{m(s, s^\infty_{1n})} \left(1 + \sum_{i=1}^{\infty} \prod_{j=1}^{i} \frac{\lambda_{-1}((s^\infty_n)^j, s^\infty_{j+1n})}{m(s^\infty_{jn}, s^\infty_{j+1n})} \right) \right) \geq M(\lambda_1, (\lambda^\infty_{-1}(s'))_{s' \in S})) - \frac{1}{n} \tag{12}$$

if $M(\lambda_1, (\lambda^\infty_{-1}(s'))_{s' \in S}))$ is finite and such that the left hand side in (12) is $\geq n$ if $M(\lambda_1, (\lambda^\infty_{-1}(s'))_{s' \in S})) = +\infty$. Then we have

$$\begin{aligned} 1 + \sup_{s' \in S} \frac{\lambda(s,s')}{m(s,s')} \cdot (J^*(s') + \varepsilon(s,s')) &\geq 1 + \left(\frac{\lambda_1(s, s^\infty_{1n})}{m(s, s^\infty_{1n})} \left(J^*(s^\infty_{1n}) + \varepsilon(s, s^\infty_{1n}) \right) \right) \\ &\geq M(\lambda_1, (\lambda^\infty_{-1}(s'))_{s' \in S})) - \frac{1}{n} \end{aligned}$$

where the last inequality follws from (12) and (11) if $M(\lambda_1, (\lambda^\infty_{-1}(s'))_{s' \in S}))$ is finite. The case $M(\lambda_1, (\lambda^\infty_{-1}(s'))_{s' \in S})) = +\infty$ is similar. Since this holds for any n, this proves (10). Thus

$$J^*(s) \leq 1 + \sup_{s' \in S} \frac{\lambda(s,s')}{m(s,s')} \cdot J^*(s') + \varepsilon.$$

Now, by the definition of the infimum, we have for some sequence of λ_n:

$$T(J^*)(s) = \lim_{n \to \infty} \left[1 + \sup_{s' \in S} \frac{\lambda_n(s,s')}{m(s,s')} \cdot J^*(s') \right].$$

Thus, for this sequence:

$$J^*(s) \leq 1 + \lim_{n\to\infty} \sup_{s' \in S} \frac{\lambda_n(s,s')}{m(s,s')} \cdot J^*(s') + \varepsilon.$$

This yields by using the previous expression for $T(J^*)(s)$

$$J^*(s) \leq T(J^*)(s) + \varepsilon.$$

Since ε was arbitrarily chosen, we have the desired conclusion:

$$J^*(s) \leq T(J^*)(s).$$

To prove the converse inequality, $J^* \geq T(J^*)$, define for any $\lambda^\infty \in \mathcal{U}^\infty(s_0)$: $J_{\lambda^\infty}(s_0) := \sup_{s^\infty \in S_0^\infty} 1 + \sum_{i=0}^{\infty} \prod_{j=0}^{i} \frac{\lambda^\infty\left((s^\infty)^j, s_{j+1}^\infty\right)}{m\left(s_j^\infty, s_{j+1}^\infty\right)}$. It suffices to show that $J_{\lambda^\infty} \geq T(J^*)$ for arbitrary transfer patterns $\lambda^\infty \in \mathcal{U}^\infty(s_0)$. In fact, we have for each $s_0 \in S$:

$$J_{\lambda^\infty}(s_0) \geq \sup_{s^\infty \in S_0^\infty} \left[1 + \frac{\lambda^\infty(s_0, s_1^\infty)}{m(s_0, s_1^\infty)} \cdot J_{\lambda_1^\infty(s_1^\infty)}(s_1^\infty)\right] \geq 1 + \sup_{s' \in S} \frac{\lambda^\infty(s_0, s')}{m(s_0, s')} \cdot J^*(s') \geq T(J^*)(s_0)$$

with $\lambda_1^\infty(s_1^\infty) \in \mathcal{U}(s_1^\infty)$ being the restriction of λ^∞ to the histories following (s_0, s_1^∞). To see the first inequality, note that if $J_{\lambda^\infty}(s_0) < \sup_{s^\infty \in S_0^\infty} \left[1 + \frac{\lambda^\infty\left(s_0, s_1^\infty\right)}{m\left(s_0, s_1^\infty\right)} \cdot J_{\lambda_1^\infty\left(s_1^\infty\right)}(s_1^\infty)\right]$, there would exist some $\widetilde{s} \in S$ such that $J_{\lambda^\infty}(s_0) < 1 + \frac{\lambda^\infty(s_0,\widetilde{s})}{m(s_0,\widetilde{s})} \cdot J_{\lambda_1^\infty(\widetilde{s})}(\widetilde{s})$. By the definition of $J_{\lambda_1^\infty(\widetilde{s})}(\widetilde{s})$, this would imply the existence of some path $\widetilde{s}^\infty \in S_0^\infty$ with $\widetilde{s}_1^\infty = \widetilde{s}$ and $J_{\lambda^\infty}(s_0) < 1 + \sum_{i=0}^{\infty} \prod_{j=0}^{i} \frac{\lambda^\infty\left((\widetilde{s}^\infty)^j, \widetilde{s}_{j+1}^\infty\right)}{m\left(\widetilde{s}_j^\infty, \widetilde{s}_{j+1}^\infty\right)}$, contradicting the definition of $J_{\lambda^\infty}(s_0)$[15]. This proves the theorem. ■

Proposition 2

Proof. **(a)** For $\lambda(s) \in \mathcal{U}(s)$, define $T_\lambda(J) := 1 + \sup_{s' \in S} \frac{\lambda(s,s')}{m(s,s')} \cdot J(s')$ for any $s \in S$ and any extended real-valued function J. We clearly have $J^* \geq J_0 \equiv 1$. Since $J^* = T(J^*)$, we have $J^* = T^N(J^*) \geq T^N(J_0)$, which implies by the definition of J_∞ that $J^* \geq J_\infty$.

To prove the converse inequality, note that $T^N(J_0)(s)$ is finite for each s and N and that $T_{\lambda_k^*}\left(T^{k-1}(J_0)\right) = T^k(J_0)$ where $\lambda_k^*(s,s') = \frac{m(s,s')}{T^{k-1}(J_0)(s')} \cdot c_k^*(s)$ and $c_k^*(s)$ is determined by $\int_S \lambda_k^*(s,s') P(s,ds') = 1$. To see that $T_{\lambda_k^*}(T^{k-1}(J_0)) = T^k(J_0)$, note that for any other function $\widehat{\lambda}(s) \in \mathcal{U}(s)$ that differs from $\lambda_k^*(s)$ on a set of $P(s,.)-$positive measure, we have $\widehat{\lambda}(s,s') > \lambda_k^*(s,s')$ for all $s' \in A$ with $P(s,A) > 0$ and $A \in \mathcal{S}$. Thus $\sup_{s' \in S} 1 + \frac{\widehat{\lambda}(s,s')}{m(s,s')} \cdot T^{k-1}(J_0)(s') > 1 + c_k^*(s) = \sup_{s' \in S} 1 + \frac{\lambda_k^*(s,s')}{m(s,s')} \cdot T^{k-1}(J_0)(s')$ and we have $T_{\widehat{\lambda}}\left(T^{k-1}(J_0)\right)(s) >$

[15] The second and third inequality hold trivially and appear in similar form on p.73 of Bertsekas and Shreve (1978).

$T_{\lambda^*}\left(T^{k-1}\left(J_0\right)\right)(s)$. For any $\widehat{\lambda}(s) \in \mathcal{U}(s)$ that differs from $\lambda^*(s)$ on a set of $P(s,.)-$zero measure, we have that $T_{\widehat{\lambda}}\left(T^{k-1}\left(J_0\right)\right)(s) = 1 + c_k^*(s) = T_{\lambda_k^*}\left(T^{k-1}\left(J_0\right)\right)(s)$.

It suffices to prove that $J^*(s_0) \leq \lim_{k\to\infty}(T_{\lambda_k^*}...T_{\lambda_1^*})(J_0)(s_0)$. We have for any $s_0 \in S$

$$\sup_{s^\infty \in S_0^\infty} 1 + \sum_{i=0}^{\infty}\prod_{j=0}^{i}\frac{\lambda_{i-j+1}^*\left(s_j, s_{j+1}\right)}{m\left(s_j, s_{j+1}\right)} \leq \lim_{k\to\infty}\left(T_{\lambda_k^*}...T_{\lambda_1^*}\right)(J_0)(s_0). \tag{13}$$

This inequality follows from the following arguments. By definition we have that:

$$\begin{aligned}
\left(T_{\lambda_k^*}...T_{\lambda_1^*}\right)(J_0)(s_0) &= 1 + \left(\sup_{s_1\in S}\frac{\lambda_k^*\left(s_0,s_1\right)}{m\left(s_0,s_1\right)}\right.\\
&+\frac{\lambda_k^*\left(s_0,s_1\right)}{m\left(s_0,s_1\right)}\cdot\left(\sup_{s_2\in S}\frac{\lambda_{k-1}^*\left(s_1,s_2\right)}{m\left(s_1,s_2\right)}\right.\\
&+\frac{\lambda_{k-1}^*\left(s_1,s_2\right)}{m\left(s_1,s_2\right)}\cdot\left(\sup_{s_3\in S}\frac{\lambda_{k-2}^*\left(s_2,s_3\right)}{m\left(s_2,s_3\right)}+\frac{\lambda_{k-2}^*\left(s_2,s_3\right)}{m\left(s_2,s_3\right)}\cdot\right.\\
&\left.\left.\left....\cdot\sup_{s_{k-1}\in S}\left(\frac{\lambda_2^*\left(s_{k-2},s_{k-1}\right)}{m\left(s_{k-2},s_{k-1}\right)}+\frac{\lambda_2^*\left(s_{k-2},s_{k-1}\right)}{m\left(s_{k-2},s_{k-1}\right)}\cdot\sup_{s_k\in S}\frac{\lambda_1^*\left(s_{k-1},s_k\right)}{m\left(s_{k-1},s_k\right)}\right)...\right)\right)\right).
\end{aligned}$$

Fix some $\widetilde{s}^k \in \{s_0\}\times S_1 \times ... \times S_k$ and choose $s_i = \widetilde{s}_i$ for $i = 0, ..., k$ history-independent in the supremum problems on the right hand side of the previous equality. We then obtain:

$$1 + \sum_{i=0}^{k-1}\prod_{j=0}^{i}\frac{\lambda_{i-j+1}^*\left(\widetilde{s}_j, \widetilde{s}_{j+1}\right)}{m\left(\widetilde{s}_j, \widetilde{s}_{j+1}\right)} \leq \left(T_{\lambda_k^*}...T_{\lambda_1^*}\right)(J_0)(s_0).$$

Thus it follows that for any $\widetilde{s}^\infty \in S_0^\infty$:

$$1 + \sum_{i=0}^{\infty}\prod_{j=0}^{i}\frac{\lambda_{i-j+1}^*\left(\widetilde{s}_j, \widetilde{s}_{j+1}\right)}{m\left(\widetilde{s}_j, \widetilde{s}_{j+1}\right)} \leq \lim_{k\to\infty}\left(T_{\lambda_k^*}...T_{\lambda_1^*}\right)(J_0)(s_0).$$

This proves (13). Together with the inequality $J^*(s_0) \leq 1 + \sup_{\widetilde{s}^\infty\in S_0^\infty}\sum_{i=0}^{\infty}\prod_{j=0}^{i}\frac{\lambda_{i-j+1}^*(\widetilde{s}_j,\widetilde{s}_{j+1})}{m(\widetilde{s}_j,\widetilde{s}_{j+1})}$, this proves the result.

(b) Note that $T^N(J_0)$ is $\mathcal{S}-$measurable by induction [using Theorem 18.7 in Aliprantis and Border (1999)]. Thus J^* is $\mathcal{S}-$measurable as the limit. ■

Proposition 3

Proof. Recall that $J^* = T(J^*)$ means that $J^*(s) = 1 + \inf_{\lambda(s)\in\mathcal{U}(s)}\sup_{s'\in S}\frac{\lambda(s,s')}{m(s,s')}\cdot J^*(s')$. Thus $J^*(s) < \infty$ for some $s \in S$ if and only if $J^*(s) < \infty$ for all $s \in S$. To see this, notice that $J^*(s) = \infty$ for some $s \in S$ implies the existence of $A \in \mathcal{S}$ with $P(s, A) = 1$ such that $J^*(s') = \infty$ for each $s' \in A$. From our assumption about the transition probabilities this implies $P(\widetilde{s}, A) = 1$ for each $\widetilde{s} \in S$. But then it immediately follows from (3) that $J^*(\widetilde{s}) =$

∞ for each $\widetilde{s} \in S$. Note that in this case any choice of $\lambda(s) \in \mathcal{U}(s)$ satisfies the condition in the proposition. In the other case $\lambda^*(s, s') = \frac{m(s,s')}{J^*(s')} \cdot c(s)$ with $\int_S \lambda(s, s') P(s, ds') = 1$ attains the minimum in the infimum part of recursive the minimax problem. This integral is well-defined since J^* is measurable by Proposition 2 (b) and is bounded below. We further have

$$J^*(s_0) = 1 + \sup_{s_1 \in S} \frac{\lambda^*(s_0, s_1)}{m(s_0, s_1)} \cdot J^*(s_1) = 1 + \sup_{s_1 \in S} \frac{\lambda^*(s_0, s_1)}{m(s_0, s_1)} \cdot \left(1 + \sup_{s_2 \in S} \frac{\lambda^*(s_1, s_2)}{m(s_1, s_2)} \cdot J^*(s_2)\right)$$

$$= 1 + \sup_{s_1 \in S} \frac{\lambda^*(s_0, s_1)}{m(s_0, s_1)} \cdot \left(1 + \sup_{s_2 \in S} \frac{\lambda^*(s_1, s_2)}{m(s_1, s_2)} \cdot \left(... \left(1 + \sup_{s_n \in S} \frac{\lambda^*(s_{n-1}, s_n)}{m(s_{n-1}, s_n)} \cdot J^*(s_n)\right) ...\right)\right)$$

for each $n \in \mathbb{N}$. Since $J^*(s) \geq 1$ for each $s \in S$, we obtain for each fixed path $s^\infty \in S_0^\infty$

$$J^*(s_0) \geq 1 + \frac{\lambda^*(s_0^\infty, s_1^\infty)}{m(s_0^\infty, s_1^\infty)} \cdot \left(1 + \frac{\lambda^*(s_1^\infty, s_2^\infty)}{m(s_1^\infty, s_2^\infty)} \cdot \left(... \left(1 + \frac{\lambda^*\left(s_{n-1}^\infty, s_n^\infty\right)}{m\left(s_{n-1}^\infty, s_n^\infty\right)}\right) ...\right)\right)$$

which is the same as

$$J^*(s_0) \geq 1 + \sum_{i=0}^{n-1} \prod_{j=0}^{i} \frac{\lambda^*\left(\left(s_j^\infty\right), s_{j+1}^\infty\right)}{m\left(s_j^\infty, s_{j+1}^\infty\right)}$$

Since n and the path $s^\infty \in S_0^\infty$ were chosen arbitrarily, the equation in the proposition follows as an inequality $\geq$. But the reverse inequality holds by definition, so that the proposition follows. ■

Proposition 4

Proof. Note that $J^*(s) = 1 + c(s)$ for all $s \in S$ for the $c(s)$ defined in the proof of Proposition 3. Thus we have:

$$\int_S \frac{m(s, s')}{J^*(s')} \cdot c(s) P(s, ds') = 1$$

$$\Longleftrightarrow \int_S m(s, s') \cdot \frac{J^*(s)}{J^*(s')} P(s, ds') = \frac{J^*(s)}{c(s)} = \frac{1}{c(s)} + 1 > 1.$$

The inequality is thus equivalent to

$$\int_S m(s, s') \cdot \frac{1}{J^*(s')} P(s, ds') > \frac{1}{J^*(s)}.$$

Therefore $\eta(s) := \frac{1}{J^*(s)}$ does the job. ■

Theorem 2

Proof. We first prove the "only if"-part of (a) and (b)[16]: Note that since the allocation is interior and the function η is bounded above, for α sufficiently small and all $s \in S$, the function

$$h(s,\alpha) = \int_S u\left(c^y(s) - \alpha\eta(s), c^o(s,s') + \alpha\eta(s')\right) P(s,ds')$$

is well defined. We show now that $h(s,\alpha) > h(s,0)$ for each $s \in S$ and some $\alpha > 0$. The reallocation that gives $c^o(s_0) + \alpha\eta(s_0)$ to each initial old person in starting state s_0 and gives to each two period lived person born in state s, $c^y(s) - \alpha\eta(s)$ when young and $c^o(s,s') + \alpha\eta(s')$ when old is interim Pareto improving for α sufficiently small.

Notice that the derivative of $h(s,\alpha)$ with respect to α is:[17]

$$h'(s,\alpha) = \int_S u_1(s,s',\alpha)(-\eta(s)) P(s,ds') + \int_S u_2(s,s',\alpha)\eta(s') P(s,ds').$$

The second order derivative can be computed with similar arguments and is, as a continuous function of α, bounded below by some negative number for values of α in a compact intervall around zero. We now have by a second order expansion for values of α in this intervall:

$$\frac{h(s,\alpha) - h(s,0)}{\alpha} \geq h'(s,0) - H\alpha$$

for some $H > 0$. By (7) and the continuity assumptions and the compactness of S, $\int_S u_2(s,s')\eta(s') P(s,ds') - \int_S u_1(s,s')\eta(s) P(s,ds') > \delta > 0$ for all $s \in S$. Thus $h(s,\alpha) > h(s,0)$ for α sufficiently small and the result follows.

The "if"-part of (a) follows since by induction and Lemma 9.5 in Stokey and Lucas with Prescott (1989) $c_k^*(s)$ from the proof of Proposition 2 (a) is a continuous function under Assumption 1. Thus $T^N(J_0)(s)$ is continuous and $J^*(s)$ is - by Proposition 2 (b) - lower semicontinuous as the monotone limit of continuous functions. Hence $\frac{1}{J^*(s)}$ is upper semicontinuous.

Now we turn to the "if"-part of (b): We show that $J^*(s)$ and hence $\eta(s)$ in (6) are continuous on S under the continuity assumptions on allocations and transition probabilities. To see this, note that the continuity of J^* is equivalent to the continuity of $c(s)$, which is in fact equivalent to the continuity of $\int_S \frac{m(s,s')}{J^*(s')} P(s,ds')$, where we can only use that $J^*(s')$ is a measurable function by Proposition 2 (b). We have

$$\left| \int_S \frac{m(s_n,s')}{J^*(s')} P(s_n,ds') - \int_S \frac{m(s,s')}{J^*(s')} P(s,ds') \right|$$

[16] Demange and Laroque (2000) have a similar proof of their Theorem 1.

[17] That we are allowed to change the order of differentiation and integration when we differentiate with respect to α follows from a standard dominated convergence argument, since given that η is bounded above and the allocation is interior, the derivative with respect to α of the integrand is pointwise bounded by a constant, which is integrable with respect to the probability measure $P(s,ds')$.

$$\leq \int_S |m(s_n, s') - m(s, s')| \frac{1}{J^*(s')} P(s_n, ds') + \left| \int_S \frac{m(s, s')}{J^*(s')} P(s_n, ds') - \int_S \frac{m(s, s')}{J^*(s')} P(s, ds') \right|$$

The first summand converges to zero as $n \to \infty$ because $m(s, s')$ is a uniformly continuous function on $S \times S$ and $\frac{1}{J^*(s')}$ is bounded above, the second summand converges to zero by Assumption 2 on $P(s, \cdot)$ [see Exercise 11.2 on p.335 in Stokey and Lucas with Prescott (1989)]. Thus $J^*(s)$ and hence $\eta(s)$ is a continuous function. ■

Corollary 1

Proof. If (8) holds, we have that (7) holds for $\eta(s) = \frac{1}{u'(c^y(s))}$. Conversely, if (7) holds for some continuous η, by defining $\alpha(s) = \eta(s) \cdot u'(c^y(s))$, (7) is equivalent to

$$\int_S \frac{v'(c^o(s'))}{u'(c^y(s'))} \cdot \alpha(s') P(ds') > \alpha(s) \quad \text{for all } s \in S$$

Since the left hand side of this inequality is independent of s, $\int_S \frac{v'(c^o(s'))}{u'(c^y(s'))} \cdot \alpha(s') P(ds') > \overline{\alpha}$, where $\overline{\alpha} := \max_{s \in S} \alpha(s)$, holds. But then also

$$\int_S \frac{v'(c^o(s'))}{u'(c^y(s'))} \cdot \overline{\alpha} P(ds') > \overline{\alpha}$$

holds, which is equivalent to (8). ■

Theorem 3

We first show the "only if"-part of the theorem, namely that if the original stationary (autarky) allocation is inefficient, then there exists a stationary monetary equilibrium. This is done in two steps. First, we show that for an artificial economy with strictly positive dividends, there exists a stationary equilibrium (Lemma 1). Second, we show that by letting the dividends go to zero, the limit is the desired monetary equilibrium *if the original equilibrium was inefficient* (Lemma 2).[18]

Let $d > 0$ be the dividend that is paid independent of the history at each date-event. Let $d_n \downarrow 0$ be a given sequence of dividends that converges to zero. As additional notation, define $\underline{c}^o := \min_{s \in S} c^o(s)$, $\overline{c}^o := \max_{s \in S} c^o(s)$, $\underline{c}^y := \min_{s \in S} c^y(s)$ and $\overline{c}^y := \max_{s \in S} c^y(s)$. $\|\cdot\|_\infty$ denotes the supremum norm on the space of continuous functions $C(S)$. We first show that for a fixed d_n, a stationary solution of the Euler equation exists:

Lemma 1 *There exists a positive continuous function $p_n(s)$ such that*

$$u'(c^y(s) - p_n(s)) p_n(s) = \int_S v'(c^o(s') + d_n + p_n(s')) \cdot (d_n + p_n(s')) P(s, ds'). \quad (14)$$

[18] In the context with a finite state space, Aiyagari and Peled (1991) use the same basic idea, but the details of the arguments are different.

Proof. The proof of existence of such a function p_n is a fixed point problem. We apply Schauder's fixed point theorem to an operator $\widetilde{T}_n$ that we define by:

$$\widetilde{T}_n = T_2 \circ T_{1n},$$

where

$$\begin{aligned} T_{1n} &: \widetilde{C}(S) \to \widetilde{\widetilde{C}}(S) \\ T_{1n}(p)(s) &= \int_S v'\left(c^o(s') + d_n + p(s')\right) \cdot \left(d_n + p(s')\right) P(s, ds'). \end{aligned}$$

Here, $\widetilde{C}(S) = \{f \in C(S) \,|0 \leq f(s) \leq c^y(s)\}$ and $\widetilde{\widetilde{C}}(S) = \{f \in C(S) \,|0 \leq f(s) \leq v'(\underline{c}^o) \cdot (d_1 + c^y(s))\}$. Clearly, from our assumptions, the function $T_{1n}(p)$ is continuous and $0 \leq T_{1n}(p)(s) \leq v'(\underline{c}^o) \cdot (d_1 + c^y(s))$. Thus the mapping is well defined. Also, define the mapping:

$$\begin{aligned} T_2 &: \widetilde{\widetilde{C}}(S) \to C_+(S) \\ u'\left(c^y(s) - T_2(h)(s)\right) \cdot T_2(h)(s) &= h(s) \end{aligned}$$

Note that the mapping $f(x) = u'(y-x)\,x$ is monotone and thus with the Inada condition invertible for every $z = f(x)$ and that $T_2(h)(s_n) \to T_2(h)(s)$ when $s_n \to s$. Thus also T_2 is well defined.

We next show that $\widetilde{T}_n$ is a continuous mapping, i.e. $\left\|\widetilde{T}_n(p_m) - \widetilde{T}_n(p)\right\|_\infty \to 0$ when $\|p_m - p\|_\infty \to 0$. We do this by showing that T_{1n} and T_2 are both continuous mappings.

First, T_{1n} is continuous: Let $\|p_m - p\|_\infty \to 0$. We have

$$\|T_{1n}(p_m) - T_{1n}(p)\|_\infty$$

$$= \sup_{s\in S}\left|\int_S \left[v'\left(c^o(s') + d_n + p_m(s')\right)\left(d_n + p_m(s')\right) - v'\left(c^o(s') + d_n + p(s')\right)\left(d_n + p(s')\right)\right] P(s, ds')\right|$$

$$\leq \sup_{s\in S}\int_S \left|v'\left(c^o(s') + d_n + p_m(s')\right)\left(d_n + p_m(s')\right) - v'\left(c^o(s') + d_n + p(s')\right)\left(d_n + p(s')\right)\right| P(s, ds').$$

Note that the function $g(x, s') := v'(c^o(s') + d_n + x)\cdot(d_n + x)$ is continuous on $[0, \overline{c}^y] \times S$ and thus uniformly continuous on this compact set. Thus for each $\varepsilon > 0$, there is a $\delta > 0$ such that $|g(p_m(s'), s') - g(p(s'), s')| < \varepsilon$ for all $s' \in S$ whenever $\|p_m - p\|_\infty < \delta$. From the definition of g it thus follows that $\|T_{1n}(p_m) - T_{1n}(p)\|_\infty < \varepsilon$ when $\|p_m - p\|_\infty < \delta$, which proves the result.

Second, T_2 is continuous: Consider the function $\widetilde{g}(s, y)$ (implicitly) defined by: $u'(c^y(s) - \widetilde{g}(s, y))\,\widetilde{g}(s, y) = y$. $\widetilde{g}$ is continuous on $S \times \left[0, \widetilde{A}\right]$, where $\widetilde{A} := \max_{s\in S} v'(\underline{c}^o) \cdot (d_1 + c^y(s))$. Thus it is uniformly continuous on this compact set, which implies that for each $\varepsilon > 0$ there exists a $\delta > 0$ such that $\left|\widetilde{g}\left(s, \widehat{h}(s)\right) - \widetilde{g}(s, h(s))\right| < \varepsilon$ for each $s \in S$

whenever $\left\|\widehat{h}-h\right\|_{\infty}<\delta$. This implies that T_2 is continuous on $\widetilde{\widetilde{C}}(S)$.

We now show that $\widetilde{T}_n\left(\widetilde{C}(S)\right)\subseteq\widetilde{C}(S)$ for each n. Note that given $T_{1n}(p)(s)$, $\widetilde{T}_n(p)(s)$ is determined by $u'\left(c^y(s)-\widetilde{T}_n(p)(s)\right)\cdot\widetilde{T}_n(p)(s)=T_{1n}(p)(s)$, which clearly implies $0\leq\widetilde{T}_n(p)(s)\leq c^y(s)$ and thus $\widetilde{T}_n\left(\widetilde{C}(S)\right)\subseteq\widetilde{C}(S)$.

Clearly, $\mathcal{F}:=\bigcup_{n=1}^{\infty}\widetilde{T}_n\left(\widetilde{C}(S)\right)\subseteq\widetilde{C}(S)$, and thus the set $\mathcal{F}$ is pointwise bounded. We want to show that the set $\mathcal{F}$ is also equicontinuous. By the Arzela-Ascoli Theorem, this implies together with the fact that $\mathcal{F}$ is pointwise bounded that $\mathcal{F}$ is relatively compact.

To show that $\mathcal{F}$ is equicontinuous, we have to show that for each $\varepsilon>0$ there exists a $\delta>0$ such that $d(s,\widetilde{s})<\delta$ implies $|p(s)-p(\widetilde{s})|<\varepsilon$ for each $p\in\mathcal{F}$. Let $p=T_2\circ T_{1n}(\widetilde{p})$ for some $\widetilde{p}\in\widetilde{C}(S)$ and some n be given. For $s,\widetilde{s}\in S$ we have

$$\begin{aligned}&\left|T_{1n}(\widetilde{p})(s)-T_{1n}(\widetilde{p})(\widetilde{s})\right|\\ =&\left|\int_S v'\left(c^o(s')+d_n+\widetilde{p}(s')\right)\cdot\left(d_n+\widetilde{p}(s')\right)\left(P(s,ds')-P(\widetilde{s},ds')\right)\right|\\ &\leq\widetilde{A}\left\|P(s,ds')-P(\widetilde{s},ds')\right\|_{TV}\end{aligned}$$

By Assumption 3, the mapping $P(\cdot,ds')$ from S to $\mathcal{P}(S)$ is continuous when $\mathcal{P}(S)$ is endowed with the variation norm. Since S is compact, the mapping is uniformly continuous and so for each $\varepsilon_1>0$ there exists a $\delta_1>0$ such that $d(s,\widetilde{s})<\delta_1$ implies $\left|T_{1n}(\widetilde{p})(s)-T_{1n}(\widetilde{p})(\widetilde{s})\right|<\frac{\sqrt{\varepsilon_1}}{2}$ for each $\widetilde{p}\in\widetilde{C}(S)$ and each n.

Next consider the mapping f defined by $u'(z-f(y,z))\,f(y,z)=y$ on $\left[0,\widetilde{A}\right]\times[\underline{c}^y,\overline{c}^y]$. This mapping is uniformly continuous. Using the definition of T_2, for each $\varepsilon>0$ there exists a $\varepsilon_1>0$ such that $\left|c^y(s)-c^y(\widetilde{s})\right|^2+\left|T_{1n}(\widetilde{p})(s)-T_{1n}(\widetilde{p})(\widetilde{s})\right|^2<\varepsilon_1$ implies $|p(s)-p(\widetilde{s})|<\varepsilon$. Note that $\left|c^y(s)-c^y(\widetilde{s})\right|^2+\left|T_{1n}(\widetilde{p})(s)-T_{1n}(\widetilde{p})(\widetilde{s})\right|^2<\varepsilon_1$ holds especially if $\left|c^y(s)-c^y(\widetilde{s})\right|<\frac{\sqrt{\varepsilon_1}}{2}$ and $\left|T_{1n}(\widetilde{p})(s)-T_{1n}(\widetilde{p})(\widetilde{s})\right|<\frac{\sqrt{\varepsilon_1}}{2}$.

Finally, by assumption, $c^y(s)$ is a uniformly continuous function, so for each $\varepsilon_1>0$ there exists a $\delta_2>0$ such that $d(s,\widetilde{s})<\delta_2$ implies $\left|c^y(s)-c^y(\widetilde{s})\right|<\frac{\sqrt{\varepsilon_1}}{2}$. Define $\delta:=\min\{\delta_1,\delta_2\}$. Thus overall, we have that $d(s,\widetilde{s})<\delta$ implies $\left|T_{1n}(\widetilde{p})(s)-T_{1n}(\widetilde{p})(\widetilde{s})\right|<\frac{\sqrt{\varepsilon_1}}{2}$ for each $\widetilde{p}\in\widetilde{C}(S)$ and each n and $\left|c^y(s)-c^y(\widetilde{s})\right|<\frac{\sqrt{\varepsilon_1}}{2}$, and thus that
$\left|T_2\circ T_{1n}(\widetilde{p})(s)-T_2\circ T_{1n}(\widetilde{p})(\widetilde{s})\right|<\varepsilon$ for each $\widetilde{p}\in\widetilde{C}(S)$ and each n, proving that $\mathcal{F}$ is equicontinuous.

The operator $\widetilde{T}_n$ thus satisfies all the requirements of the Schauder fixed point theorem as stated in Heuser (2004), Theorem 230.4 (b) and and we have a sequence of fixed points $p_n=\widetilde{T_n}(p_n)$. ■

Lemma 17.5 in Stokey and Lucas with Prescott (1989) shows equicontinuity for the image of an operator (to be able to apply Schauder's fixed point theorem) that corresponds to T_{1n} in our framework and uses on the transition probability $P(s,ds')$ our Assumption 3. The case considerd here is however simpler, since our expression does not depend on s, only on s'. Apart from this, the construction in Stokey and Lucas with Prescott (1989)

differs in the details.
From (14), all p_n are strictly positive. Since $\{p_n\} \subseteq \mathcal{F}$ and $\mathcal{F}$ is relatively compact, $\{p_n\}$ must have a convergent subsequence $\{p_{n_k}\}$. To simplify notation, we denote this subsequence by $\{p_k\}$. Let p^* be its limit point. Note that by dominated convergence, (14) with $d_n = 0$ also holds for p^*, so that by our assumption that each transition probability $P(s,\cdot)$ has full support, either $p^* = 0$ or p^* is strictly positive. A strictly positive p^* is clearly a monetary equilibrium.

We now turn to the second step of our argument and let the dividends go to zero.

Lemma 2 *Let a continuous stationary allocation be given. If the allocation is interim Pareto inefficient there exists a monetary equilibrium associated with it.*

Proof. Suppose the original stationary (autarky) allocation $\{c^y(s), c^o(s)\}_{s\in S}$ is interim Pareto inefficient and $p^* = 0$. By Theorem 2 this implies the existence of strictly positive continuous function η such that

$$\int_S \frac{v'(c^o(s'))}{u'(c^y(s))}\eta(s') P(s, ds') > \eta(s) \quad \text{for each } s \in S.$$

Since $p^* = 0$, we have that $\|p_k\|_\infty \to 0$. Assume w.l.o.g. that $\|p_k\|_\infty \downarrow 0$. Observe that $\int_S \frac{v'(c^o(s')+p_k(s')+d_k)}{u'(c^y(s)-p_k(s))}\eta(s') P(s,ds') \geq \int_S \frac{v'\left(c^o(s')+\|p_k\|_\infty+d_k\right)}{u'\left(c^y(s)-\|p_k\|_\infty\right)}\eta(s') P(s,ds')$ and both are continuous functions in s.[19]
$\int_S \frac{v'\left(c^o(s')+\|p_k\|_\infty+d_k\right)}{u'\left(c^y(s)-\|p_k\|_\infty\right)}\eta(s') P(s,ds')$ converges monotonically to the continuous function $\int_S \frac{v'(c^o(s'))}{u'(c^y(s))}\eta(s') P(s,ds')$ on the compact set S. Thus by Dini's lemma [see Theorem 2.62 in Aliprantis and Border (1999)], convergence is uniform. Further by continuity and compactness $\min_{s\in S} \int_S \frac{v'(c^o(s'))}{u'(c^y(s))}\eta(s') P(s,ds') - \eta(s) > 0$. Thus for k sufficiently large,

$$\int_S \frac{v'(c^o(s') + p_k(s') + d_k)}{u'(c^y(s) - p_k(s))}\eta(s') P(s, ds') > \eta(s) \quad \text{for each } s \in S. \tag{15}$$

By Theorem 2, this implies that the stationary allocation $\{c^y(s) - p_k(s), c^o(s) + p_k(s) + d_k\}_{s\in S}$ is inefficient. However, an equilibrium with an asset that pays strictly positive dividends at each date is always interim Pareto efficient (see Lemma 4). This gives a contradiction to (15) and shows that we cannot have $p^* = 0$ if $\{c^y(s), c^o(s)\}_{s\in S}$ is interim Pareto inefficient. ■

We now state the much easier converse to this result, the "if"-part of Theorem 3:

Lemma 3 *Let a continuous stationary allocation be given. The allocation is interim Pareto inefficient if a monetary equilibrium associated with it exists.*

Proof. This follows from the strict concavity of the utility function and the strict positivity of p as in the corresponding Proposition on p.282 of Manuelli (1990). ■

[19]Note that $\|p_k\| < \underline{c}^y$ for k sufficiently large, so that $u'(c^y(s) - \|p_k\|_\infty)$ is well defined.

The following lemma shows that any allocation satisfying (14) and any allocation in a monetary equilibrium is efficient. The logic behind it is similar to Proposition 4 (a) in Barbie, Hagedorn and Kaul (2007).

Lemma 4 *Any continuous stationary allocation* $\{c^y(s) - p(s), c^o(s) + p(s) + d\}_{s \in S}$ *with* $d \geq 0$ *such that* $u'(c^y(s) - p(s))\, p(s) = \int_S v'(c^o(s') + d + p(s')) \cdot (d + p(s'))\, P(s, ds')$ *holds is interim Pareto efficient.*

Proof. Define $\widetilde{m}(s, s') := \frac{v'(c^o(s')+d+p(s'))}{u'(c^y(s)-p(s))}$. We have $\int_S \widetilde{m}(s, s')\, p(s')\, P(s, ds') \leq p(s)$. Define $\lambda^*(s, s') = \widetilde{m}(s, s') \frac{p(s')}{p(s)} \alpha(s)$ with $\int_S \lambda^*(s, s')\, P(s, ds') = 1$. Clearly, $\alpha(s) \geq 1$. We have for any path s^∞

$$\sum_{i=0}^{\infty} \prod_{j=0}^{i} \frac{\lambda^*\left(s_j^\infty, s_{j+1}^\infty\right)}{\widetilde{m}\left(s_j^\infty, s_{j+1}^\infty\right)} = \sum_{i=0}^{\infty} \frac{p\left(s_{i+1}^\infty\right)}{p\left(s_0^\infty\right)} \cdot \prod_{j=0}^{i} \alpha\left(s_j^\infty\right) = \infty. \tag{16}$$

For any family $\mu^\infty \in \mathcal{U}^\infty = \prod_{s^t} \mathcal{U}(s^t)$ such that $\mu^\infty(s^t) \in \mathcal{U}(s^t)$ for each histories s^t, we have that $\mu^\infty(s^t, s') \geq \lambda^*(s_t, s')$ for each history s^t and some $s' \in S$. Together with (16) this implies that Proposition 1 can never hold for any $\mu^\infty \in \mathcal{U}^\infty$, proving that $\{c^y(s) - p(s), c^o(s) + p(s) + d\}_{s \in S}$ is interim Pareto efficient. ■

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