Bubbly Markov Equilibria*

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Abstract

Bubbly Markov Equilibria (BME) are recursive equilibria on the natural state space which admit a non-trivial bubble. The present paper studies the existence and properties of BME in a general class of overlapping generations (OLG) economies with capital accumulation and stochastic production shocks. Using methods from functional analysis, we develop a general approach to construct Markov equilibria and provide necessary and sufficient conditions for these equilibria to be bubbly. Our main result shows that a BME exists whenever the bubbleless equilibrium is Pareto inefficient either due to overaccumulation of capital or inefficient risk-sharing between generations.

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Introduction

A bubble is an intrinsically worthless asset which trades at a positive price such as fiat money, governmental debt, or a bond that never pays any dividends. The emergence of such a bubble has two important macroeconomic effects. First, it affects the formation of capital by providing an alternative investment opportunity to investors. Second, it creates an additional insurance possibility which affects the risk sharing arrangements among consumers.

The present paper studies the existence and properties of bubbly equilibria in a unifying framework that incorporates both of these effects as well as their mutual interactions. To the best of our knowledge, we are the first to offer a comprehensive analysis of this type. In order to account for the *investment effect* of bubbles, we place our study in the class of overlapping generations (OLG) models with production and endogenous capital accumulation. To capture the *risk sharing effect*, we use a stochastic setup with exogenous random production shocks. Finally, we include a dividend paying asset in our model. An asset bubble corresponds to the limiting case where dividends are zero but the price of the asset remains strictly positive.

With the previous features, our setup encompasses the case of a deterministic production economy studied in Tirole (1985) as well as stochastic models with pure exchange as in Manuelli (1990), Aiyagari & Peled (1991), Barbie & Kaul (2015) or Magill & Quinzii (2003). By construction, these studies either neglect the investment or the risk sharing effect of bubbles. In this sense, our framework contains these models as special cases and we will discuss which role the previous existence results play in our extended setup.

The stochastic OLG model with production has been studied in Wang (1993, 1994) and, more recently, in Morand & Reffett (2007), McGovern et al. (2013), and Hillebrand (2014). All these studies focus on a particular class of equilibria in which the equilibrium variables are determined by time-invariant mappings on the minimal or 'natural' state space. Following Kübler & Polemarchakis (2004), such equilibria will be called Markov Equilibria (ME). Extending this terminology, we call a ME which admits a bubble a Bubbly Markov Equilibrium (BME). Identifying conditions under which a BME exists and characterizing its properties is the general objective of this paper.

The first part of our analysis lays out a general method to construct potentially bubbly ME. This sets the stage to establish a general existence theorem for BME in the second part. A first major obstacle to construct ME in our setup is that the pointwise fixed point methods employed in Wang (1993) are no longer applicable. For this reason, our construction is based on monotone methods from functional analysis similar to Coleman (1991, 2000), or Greenwood & Huffman (1995). This approach was successfully applied in Morand & Reffett (2007) to study bubbleless ME, and we will show how it can be extended to study BME as well. The method to be developed is also constructive and can directly be employed to compute BME numerically in applications of our results.

The goal of the second part is to provide necessary and sufficient conditions under which the ME constructed is bubbly. Our main result shows that this is the case whenever the bubbleless equilibrium is Pareto inefficient. Such an inefficiency can be the result of dynamic inefficiency as studied in Zilcha (1990), but may also be due to inefficient risk-sharing among generations. Thus, a major difference to the deterministic result in Tirole (1985) is that bubbly equilibria can emerge in stochastic economies which are dynamically efficient. We expect this result to have many promising applications such as the sustainability and optimal risk structure of governmental debt as studied in Ball, Elmendorf & Mankiw (1998) or the risk sharing properties of social security systems analyzed in Gottardi & Kübler (2011). In such applications, our construction of BME provides an algorithm for explicitly determining sustainable debt policies and the optimal risk indexation of debt returns or social security transfers.

A major challenge to establish our existence result is that it requires a workable criterion to determine when an equilibrium allocation is Pareto inefficient. Building on the results from Chattopadhyay & Gottardi (1999), a complete characterization of Pareto optimality in stochastic OLG production economies is provided in Barbie, Hagedorn & Kaul (2007). The criterion employed in this paper essentially combines their results with the recursive characterization of Pareto optimality developed in Barbie & Kaul (2015). This leads to a sort of dominant root criterion in the presence of a continuous state space, which is formulated in Barbie & Kaul (2015) for a stationary exchange economy. Similar criteria for efficiency/inefficiency are derived in Demange & Laroque (2000).

Based on this criterion, we establish our existence result by constructing a sequence of economies with a dividend-paying asset whose dividends converge to zero. Each such economy is known to have only efficient ME. Under some additional restrictions, the limiting ME of the benchmark economy is also efficient. Thus, a BME necessarily exists whenever the bubbleless equilibrium is known to be Pareto inefficient.

The paper is organized as follows. Section 1 introduces the model and defines the concept of a Markov equilibrium. Section 2 develops a general approach to construct these equilibria. Section 3 contains the main results which state necessary and sufficient conditions under which the ME constructed is bubbly. Section 4 concludes; technical proofs and derivations are relegated to the Mathematical Appendices A and B.

1 The Model

This section introduces the structure and assumptions of the basic model and formalizes the concept of a Markov equilibrium which will be at the core of the subsequent analysis.

1.1 Production sector

The production side is represented by a single firm which operates a linear homogeneous technology to produce an all-purpose output commodity using capital and labor as inputs. In addition, production in period t is subjected to an exogenous random production shock $\theta_t > 0$. At equilibrium, labor supply will be constant and normalized to unity. Given the shock, the intensive form production function $f : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ determines production output y_t in period t from the existing stock of capital $k_t \geq 0$ as

$$y_t = \theta_t f(k_t).$$

As in Wang (1993), shocks are i.i.d. over time with (marginal) distribution ν supported on the compact set $\Theta \subset \mathbb{R}_{++}$. Let $\theta_{\min}$ denote the minimal and $\theta_{\max}$ the maximal realization of the shock. The formal arguments in Section 3 assume that Θ is a finite set. The process $\{\theta_t\}_{t\geq 0}$ induces a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ on which all random variables are defined and a filtration $\{\mathcal{F}_t\}_{t\geq 0}$ such that θ_t is $\mathcal{F}_t$-measurable. Throughout, the notion of an adapted stochastic process $\{\xi_t\}_{t\geq 0}$ refers to this filtration and implies that each ξ_t can depend only on random variables θ_n, $n \leq t$. Moreover, $\mathbb{E}_t[\cdot] := \mathbb{E}[\cdot|\mathcal{F}_t]$ is the conditional expectations operator.

The following restrictions on f are standard and will be imposed throughout the paper.

Assumption 1
The map $f : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ is C^2 with derivatives $f'' < 0 < f'$ and $\lim_{k\to 0} f'(k) = \infty$. Moreover, there exists an upper bound $\bar{k} > 0$ such that $\theta_{\max} f(k) < k$ whenever $k > \bar{k}$.

Market clearing and profit maximizing behavior imply that the equilibrium wage w_t and capital return r_t are determined by the capital stock $k_t > 0$ and the shock as $\theta_t \in \Theta$ as

$$w_t = W(k_t, \theta_t) := \theta_t[f(k_t) - k_t f'(k_t)] \tag{1a}$$
$$r_t = R(k_t, \theta_t) := \theta_t f'(k_t). \tag{1b}$$

Denote by $E_\phi(z) := |\frac{z\phi'(z)}{\phi(z)}|$, $z \in \mathbb{R}$ the (absolute) elasticity of a differentiable function $\phi \neq 0$. Below we will occasionally impose the following additional restrictions on f:

$$(T1)\, E_{f'} \leq 1 \qquad (T2)\, 2E_{f'} \geq 1.$$

As (T1) is equivalent to $k \mapsto kR(k, \theta)$ being weakly increasing, this restriction is known as 'capital income monotonicity' and often imposed in the literature, cf. Wang (1993), Hauenschild (2002), and others. It holds in the Cobb-Douglas case $f(k) = k^\alpha$ for $0 < \alpha < 1$ and also for CES technologies $f(k) = [1 - a + ak^\varrho]^{\frac{1}{\varrho}}$ where $0 < a < 1$ and $0 < \varrho < 1$. The second restriction (T2) imposes a uniform lower bound on the elasticity of f'. In the Cobb-Douglas case, this restricts α to the empirically relevant case $\alpha \leq \frac{1}{2}$. When imposed, the previous conditions are only required to hold on the bounded set $\mathbb{K} =]0, k_{\max}]$ defined below.

1.2 Consumption sector

The consumption sector consists of overlapping generations of consumers who live for two periods. For simplicity, there is no population growth and the size of each generation is normalized to unity. Young consumers earn income from supplying one unit of labor inelastically to the labor market while old consumers earn the proceeds on their investments made during the previous period.

To transfer income to the second period of life, there are two investment possibilities available to a young consumer in period t. First, she can invest in capital to earn the uncertain capital return r_{t+1} in the next period per unit invested at time t. Second, she can invest in assets given by retradeable shares of a fruit tree (Lucas tree) which pay a constant non-random dividend $d \geq 0$ in each period. Let $p_t \geq 0$ denote the asset price per share in period $t \geq 0$. The total supply of shares is normalized to unity.

A young consumer in period t observes her labor income $w_t > 0$ and the buying price of shares $p_t \geq 0$ while taking the selling price $p_{t+1} \geq 0$ and the capital return $r_{t+1} > 0$ as given random variables in her decision. The consumer chooses the desired investments in capital s and in shares z to maximize expected lifetime utility. Assuming an additive von-Neumann Morgenstern utility function $U(c^y, c^o) = u(c^y) + v(c^o)$ over lifetime consumption, the decision problem reads:

$$\max_{z,s}\Big\{u(w_t - z\,p_t - s) + \mathbb{E}_t\big[v\big(z\,(p_{t+1} + d) + s\,r_{t+1}\big)\big]\,\Big|\, s \geq 0, z \geq 0, z\,p_t + s \leq w_t\Big\}. \quad (2)$$

Throughout, we impose the following standard restrictions on the utility functions.

Assumption 2
Both $g \in \{u, v\}$ are C^2 with derivatives satisfying $g'' < 0 < g'$ and $\lim_{c\to 0} g'(c) = \infty$.

The capital investment s_t in period t determines the capital stock k_{t+1} of the following period. Combining this with the first-order conditions of the decision problem (2), one obtains the following Euler equations which must hold in each period t at equilibrium:

$$u'(w_t - p_t - k_{t+1}) = \mathbb{E}_t\big[r_{t+1}v'(p_{t+1} + d + k_{t+1}r_{t+1})\big] \quad (3a)$$

$$u'(w_t - p_t - k_{t+1})p_t = \mathbb{E}_t\big[(p_{t+1} + d)v'(p_{t+1} + d + k_{t+1}r_{t+1})\big]. \quad (3b)$$

The following additional restrictions on v will occasionally be used in the sequel:

$$(U1)\ E_{v'} \leq 1 \quad (U2)\ E_{v'} \equiv \theta.$$

Condition (U1) is again a standard restriction also imposed in Morand & Reffett (2007) or McGovern et al. (2013). Under (U2) second-period utility v displays constant relative risk aversion, an assumption that is more restrictive but also widely used in applied macroeconomic models.

1.3 Markov Equilibria (ME)

The dividend payment $d \geq 0$ will be a key parameter in our analysis. For a given value $d \geq 0$, the economy is summarized by the list $\mathscr{E}_d = \langle u, v, f, \nu, d \rangle$ plus initial conditions for capital $k_0 > 0$ and the shock $\theta_0 \in \Theta$. Specifically, we refer to the economy $\mathscr{E} := \mathscr{E}_0$ in which dividend payments are zero as as the *benchmark economy* in our framework. Note that $\mathscr{E}$ corresponds to the economy studied in Wang (1993).

The following definition is standard and provides the most general notion of equilibrium.

Definition 1
Given initial values $k_0 > 0$ and $\theta_0 \in \Theta$, a sequential equilibrium (SE) of $\mathscr{E}_d$ is an adapted stochastic process $\{w_t, r_t, p_t, k_{t+1}\}_{t \geq 0}$ which satisfies (1a,b) and (3a,b) for all $t \geq 0$.

The induced equilibrium consumption processes can be recovered as $c_t^y = w_t - p_t - k_{t+1}$ and $c_t^o = k_t r_t + p_t + d = \theta_t f(k_t) + d - c_t^y - k_{t+1}$ for all $t \geq 0$.

In this paper, we focus on a particular class of equilibria where all equilibrium variables are determined by time-invariant functions of some state variable x_t which takes values in the state space $\mathbb{X}$. In the literature, such equilibria are called *Recursive Equilibria (RE)*. We confine ourselves to a particular class of recursive equilibria where the state variable is $x_t = (k_t, \theta_t)$. The underlying state space $\mathbb{X}$ is called the natural state space. Note that the factor price mappings W and R from (1a,b) already satisfy this property. Following the terminology of Kübler & Polemarchakis (2004), RE on the natural state space are called *Markov Equilibria* (ME). In the following definition, $\mathbb{X} \subset \mathbb{R}_{++} \times \Theta$ is assumed to be a non-empty Borel set which will be constructed explicitly in the next section.

Definition 2
A SE of $\mathscr{E}_d$ is called a Markov equilibrium (ME) on $\mathbb{X}$ if there exists measurable mappings $K_d^E : \mathbb{X} \longrightarrow \mathbb{R}_{++}$ and $P_d^E : \mathbb{X} \longrightarrow \mathbb{R}_+$ such that $k_{t+1} = K_d^E(k_t, \theta_t)$ and $p_t = P_d^E(k_t, \theta_t)$ for all $t \geq 0$ and all $x_0 = (k_0, \theta_0) \in \mathbb{X}$.

A primary goal of this paper is to study ME (K^E, P^E) of the benchmark economy $\mathscr{E} = \mathscr{E}_0$ where dividend payments are zero (we will occasionally drop the subscript if $d = 0$). In particular, we ask whether such equilibria admit a bubble, i.e., can be supported by a non-zero asset price process. Extending the previous terminology, we refer to a ME which admits a bubble as a *Bubbly Markov Equilibrium* (BME). Formally, we have

Definition 3
A ME (K^E, P^E) of $\mathscr{E}$ is called bubbly if $P^E \neq 0$ and bubbleless if $P^E = 0$.

In addition to their theoretical appeal, bubbly ME have several important applications and admit various alternative interpretations. One such application concerns the sustainability and optimal risk structure of governmental debt. Suppose in each period t, a government issues one-period bonds with unit price and (risk-indexed) return r^*_{t+1} to finance its current debt $b_t > 0$. Then, the process $\{b_t\}_{t\geq 0}$ which evolves as $b_{t+1} = r^*_{t+1} b_t$ is formally equivalent to a bubble in our previous framework. Exploiting this equivalence, the value $P^E(x_t)$ defines the maximum level of debt that is sustainable if the current fundamental state is $x_t \in \mathbb{X}$. Further, the optimal risk structure of the return offered in period t needed to sustain this maximum level is determined by the random variable

$$r^*_{t+1} := R^*(x_t, \cdot) = \frac{P^E(K^E(x_t), \cdot)}{P^E(x_t)}. \tag{4}$$

The existence of a BME is therefore equivalent to a positive equilibrium level of debt that can be sustained without further stabilization such as taxation, etc. Also note that (4) would permit to explicitly compute the Arrow-Debreu prices of risk at equilibrium.

An alternative interpretation of a BME is that of a monetary equilibrium in which a fixed quantity $M > 0$ of fiat money is exchanged between successive generations. In this case, the price $p_t > 0$ corresponds to real money balances in period t.

One can also interpret a BME as an equilibrium with a social security system in which $p_t > 0$ represents the transfers from young to old consumers in period $t \geq 0$. A particular appealing feature that follows from the Euler equation (3b) is that such a system is time consistent in the sense that no generation has an incentive to change it (see Hillebrand (2011) for an application of this concept). Thus, a BME directly implies the existence of a time-consistent Social Security system.

In the following section we show that the properties of the (unique) bubbleless ME of $\mathscr{E}$ are key to construct the state space $\mathbb{X}$ associated with *any* ME of $\mathscr{E}_d$ where $d \geq 0$.

1.4 Restricting the state space

It is shown in Wang (1993) and Hillebrand (2014) that either restriction (T1) or (U1) is sufficient for the benchmark economy $\mathscr{E}$ to possess a unique bubbleless ME. In this case, the equilibrium mappings are given by $P^E_0 \equiv 0$ and $K^E_0 = K_0 \circ W$ where $K_0 : \mathbb{R}_{++} \longrightarrow \mathbb{R}_{++}$ determines the unique solution $k = K_0(w)$ to the implicit condition

$$G_0(k, w) := u'(w - k) - \mathbb{E}_\nu[R(k, \cdot) v'(kR(k, \cdot))] = 0. \tag{5}$$

Note that the implicit function theorem implies that K_0 is C^1, strictly increasing, and $0 < K_0(w) < w$ for all $w > 0$. The capital process along the bubbleless ME evolves as

$$k_{t+1} = K^E_0(k_t, \theta_t) = K_0 \circ W(k_t, \theta_t). \tag{6}$$

Equation (6) is precisely the representation of equilibrium studied in Wang (1993). To rule out degenerate cases in which capital converges to zero with positive probability, he imposes the additional restriction $\lim_{k \searrow 0} \partial_k K^E_0(k, \theta_{\min}) > 1$, cf. Theorem 4.3 in Wang

(1993). This condition ensures existence of a lower bound $\underline{k} > 0$ such that $K^E(k, \theta) > k$ for all $\theta \in \Theta$ whenever $k \leq \underline{k}$.[1] For most of the following analysis, however, it suffices to work with a weaker condition which only excludes that capital converges to zero with probability one. Only later will the stronger requirement of Wang (1993) be necessary.

Before introducing such restrictions formally, a crucial observation is that the bubbleless ME is fully described by the map K_0 defined on the one-dimensional set $\mathbb{W} \subset \mathbb{R}_{++}$ of equilibrium wages. We will show in the next section that the same structure obtains in the bubbly case and also along any ME of $\mathscr{E}_d$, $d > 0$. In each case, the equilibrium mappings in Definition 2 take the form $P_d^E = P_d \circ W$ and $K_d^E = K_d \circ W$ with P_d and K_d defined on $\mathbb{W}$. Thus, any ME is completely described by mappings defined on a one-dimensional set $\mathbb{W}$ which we will refer to as the *reduced state space*. For this reason, the pair (P_d, K_d) will also be referred to as a ME of $\mathscr{E}_d$.

It will be convenient to impose restrictions on the reduced state space $\mathbb{W}$ rather than $\mathbb{X}$ directly. For this reason, we state the aforementioned boundary properties in terms of the wage process along the bubbleless equilibrium which evolves as

$$w_{t+1} = W_0^E(w_t, \theta_{t+1}) := W(K_0(w_t), \theta_{t+1}). \tag{7}$$

Mathematically, this representation of the equilibrium dynamics is equivalent to (6). The following assumption rules out that the wage process (7) converges to zero with probability one.

Assumption 3
The map W_0^E *defined in (7) satisfies* $\liminf_{w \searrow 0} W_0^E(w, \theta_{\max})/w > 1$.

Assumptions 1 and 3 together with continuity of W_0^E ensure that the set of fixed points of $W_0^E(\cdot, \theta_{\max})$ is non-empty and compact. Thus, defining

$$w_{\max} := \min\Big\{ w > 0 \Big| w = W_0^E(w, \theta_{\max}) \Big\} \tag{8}$$

allows us to use $\mathbb{W} :=]0, w_{\max}]$ as the reduced state space. Note that $\mathbb{W}$ is self-supporting in the sense that $w \in \mathbb{W}$ implies $W_0^E(w, \theta) \in \mathbb{W}$ for all $\theta \in \Theta$. Further, $W_0^E(\cdot, \theta_{\max})$ has $w_{\max}$ as its unique fixed point which is globally asymptotically stable on $\mathbb{W}$.[2]

Setting $k_{\max} := K_0(w_{\max})$ and $\mathbb{K} =]0, k_{\max}]$ permits to define $\mathbb{X} := \mathbb{K} \times \Theta$ as the natural state space from Definition 2 along the bubbleless ME. In the next section, we show that these choices for $\mathbb{W}$ and $\mathbb{X}$ extend to the bubbly case and any ME of $\mathscr{E}_d$ if $d > 0$. Thus, a major advantage of Assumption 3 is that it permits a bounded state space.

[1] A similar restriction is employed in Hauenschild (2002) and others. Conditions on the primitives under which these properties hold can be found in Galor & Ryder (1989) or, more recently, in Li & Lin (2012).

[2] This uniqueness property will be important to obtain several results including Theorem 1. Otherwise, we could have defined $w_{\max}$ in (8) to be the maximum fixed point of $W_0^E(\cdot, \theta_{\max})$.

Some results of Section 3 will even require that $\mathbb{W}$ and $\mathbb{X}$ can be chosen compact along the bubbleless ME. In such cases, the following stronger restriction is imposed which rules out that the wage process (7) converges to zero even with positive probability. Mathematically, this is equivalent to the condition in Wang (1993) discussed above.

Assumption 4
The map W_0^E defined in (7) satisfies $\liminf_{w \searrow 0} W_0^E(w, \theta_{\min})/w > 1$.

Assumption 4 permits to choose a value $\underline{w} > 0$ such that $W_0^E(w, \theta_{\min}) > w$ for all $w \leq \underline{w}$. Thus, $W_0^E(\cdot, \theta_{\min})$ has at least one positive fixed point. Defining

$$w_{\min} := \min\Big\{ w > 0 \Big| w = W_0^E(w, \theta_{\min}) \Big\} \tag{9}$$

ensures that $\overline{\mathbb{W}} := [w_{\min}, w_{\max}]$ is a compact self-supporting set for the dynamics (7). While this would also permit to choose the state space $\mathbb{X}$ compact along the bubbleless ME, these choices neither extend to the bubbly case nor to a ME of $\mathscr{E}_d$ if $d > 0$.

2 Constructing Markov Equilibria

The pointwise construction of ME employed in Wang (1993) and the previous section is available only in the bubbleless case. For this reason, the following sections develop a more general approach which is based on methods from functional analysis similar to Morand & Reffett (2007). It is shown in Hillebrand (2014) that this approach is equivalent to the pointwise construction in Wang (1993) in the bubbleless case. Our method permits to construct ME of the general class of economies $\mathscr{E}_d$, $d \geq 0$ introduced in the previous section. Identifying conditions under which the solution obtained for $d = 0$ defines a bubbly ME of the benchmark economy $\mathscr{E}$ then becomes a separate issue to be explored in Section 3.

The following sections throughout impose Assumptions 1, 2, and 3. Using the results from Section 1.4, define $w_{\max}$ as in (8) and the reduced state space $\mathbb{W} =]0, w_{\max}]$, $k_{\max} := K_0(w_{\max})$ by (5), $\mathbb{K} =]0, k_{\max}]$, and the natural state space $\mathbb{X} = \mathbb{K} \times \Theta$.

2.1 Defining an operator T_d

Given $d \geq 0$, the following analysis aims to construct ME of $\mathscr{E}_d$ as fixed points of an operator T_d defined on some suitably chosen function space $\mathscr{G}$. To restrict the class of candidate equilibrium functions $\mathscr{G}$, a first and crucial observation is that the current state $x_t = (k_t, \theta_t)$ enters the Euler equations (3a,b) only through the wage $w_t = W(x_t)$.[3]

[3]This property rests crucially on the i.i.d. structure of the shock process. While this will simplify the subsequent construction of ME considerably, we expect the underlying principle along with most of

Therefore, we conjecture that, similar to the bubbleless equilibrium, the mappings from Definition 2 can be written as $K_d^E = K_d \circ W$ and $P_d^E = P_d \circ W$ where $K_d : \mathbb{W} \longrightarrow \mathbb{K}$ and $P_d : \mathbb{W} \longrightarrow \mathbb{R}_+$. Under this hypothesis, the problem of determining a ME is equivalent to determining the two functions (K_d, P_d) consistent with the Euler equations (3a,b). Moreover, we will show below that any solution P_d uniquely determines the associated capital function K_d. Thus, we are essentially left to determine the function P_d. We restrict our search for this solution to the function space

$$\mathscr{G} := \left\{ P : \mathbb{W} \longrightarrow \mathbb{R}_+ \;\middle|\; \begin{array}{l} P \text{ is continuous} \\ w \mapsto P(w) \text{ is weakly increasing} \\ w \mapsto w - P(w) \text{ is weakly increasing} \end{array} \right\}. \tag{10}$$

The space $\mathscr{G}$ is endowed with the usual pointwise ordering, i.e., $P_1 \geq P_2$ ($P_1 > P_2$) iff $P_1(w) \geq P_2(w)$ ($P_1(w) > P_2(w)$) for all $w \in \mathbb{W}$.

The previous insights greatly simplify the construction of ME because they permit to reduce the problem of determining two functions (P_d^E, K_d^E) both defined on $\mathbb{X}$ to finding a single function P_d defined on the one-dimensional space $\mathbb{W}$. In the sequel we will construct P_d as a fixed point of some operator T_d defined on $\mathscr{G}$. The additional monotonicity restrictions in (10) will be necessary for this operator to be well-defined.

Let $d \geq 0$ be arbitrary but fixed. The key ingredient to construct the operator T_d are the Euler equations (3a,b). The idea is as follows: At some fixed point in time, suppose next period's asset price is determined by some function $P \in \mathscr{G}$ of next period's wage. Then, for any current state $w \in \mathbb{W}$, the current price p and capital investment k must solve the Euler equations (3a,b). Given $P \in \mathscr{G}$ and some fixed $w \in \mathbb{W}$, let

$$\begin{aligned} H^1(k, p; w, P, d) &:= u'(w - p - k) \\ &\qquad - \mathbb{E}_\nu\big[R(k,\cdot) v'\big(P(W(k,\cdot)) + d + kR(k,\cdot)\big)\big] \end{aligned} \tag{11a}$$

$$\begin{aligned} H^2(k, p; w, P, d) &:= u'(w - p - k)p \\ &\qquad - \mathbb{E}_\nu\big[\big(P(W(k,\cdot)) + d\big) v'\big(P(W(k,\cdot)) + d + kR(k,\cdot)\big)\big] \end{aligned} \tag{11b}$$

which are defined for all $0 < k < k_{\max}$ and $p \geq 0$ such that $k + p < w$. Then, for any fixed $\tilde{w} \in \mathbb{W}$, the problem is to determine $\tilde{k} \in \mathbb{K}$, and $\tilde{p} \geq 0$ such that $\tilde{k} + \tilde{p} < \tilde{w}$ and

$$H^1(\tilde{k}, \tilde{p}; \tilde{w}, P, d) = H^2(\tilde{k}, \tilde{p}; \tilde{w}, P, d) = 0. \tag{12}$$

First, consider the problem (12) for $d = 0$. For this case, we have the following result.

Lemma 2.1
In addition to Assumptions 1–3, let (T1) and (U1) be satisfied and suppose $d = 0$. Then, for any $P \in \mathscr{G}$ and $\tilde{w} \in \mathbb{W}$, there is a unique solution $\tilde{p} \geq 0$ and $\tilde{k} \in \mathbb{K}$ to (12).

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the results to carry over to more general classes of economies including correlated production shocks. Clearly, in this case the function space $\mathscr{G}$ consists of mappings defined on $\mathbb{X}$ rather than $\mathbb{W}$.

Lemma 2.1 permits to define functions $TP : \mathbb{W} \longrightarrow \mathbb{R}_+$ and $K_P : \mathbb{W} \longrightarrow \mathbb{R}_{++}$ which determine the unique solution to (12) if $d = 0$, i.e., $TP(\tilde{w}) := \tilde{p}$ and $K_P(\tilde{w}) := \tilde{k}$ for each $\tilde{w} \in \mathbb{W}$. This induces an operator T on $\mathscr{G}$ which associates with any function $P \in \mathscr{G}$ the new function $T(P) := TP$. We also denote by $K_\bullet$ the operator on $\mathscr{G}$ which assigns to $P \in \mathscr{G}$ the function K_P.[4]

The following result shows that T maps $\mathscr{G}$ into itself and establishes several additional properties. Here, the additional restrictions (T2) and (U2) are needed to ensure that T maps monotonic functions to monotonic functions.

Lemma 2.2
In addition to Assumptions 1–3, let (T1), (U1), and either (T2) or (U2) hold. Then $T : \mathscr{G} \longrightarrow \mathscr{G}$. Further, for each $P \in \mathscr{G}$ the following holds:

(i) $T(P) < \mathrm{id}_{\mathbb{W}}$, $P > 0$ *implies* $T(P) > 0$ *while* $P = 0$ *implies* $T(P) = 0$.

(ii) K_P *is continuous and increasing,* $K_P \leq K_0 < \mathrm{id}_{\mathbb{W}}$ *and* $P > 0$ *implies* $K_P < K_0$.

In a second step, consider now the problem (12) for $d > 0$. Observe that this problem is identical to the case where $d = 0$ if P is replaced by the function $\hat{P} = P + d$, i.e., $\hat{P}(w) := P(w) + d$ for all $w \in \mathbb{W}$. Clearly, $P \in \mathscr{G}$ implies $P + d \in \mathscr{G}$ for all $d \geq 0$. Thus, define for each fixed $d \geq 0$ the operator T_d on $\mathscr{G}$ as

$$T_d(P) = T(P + d). \tag{13}$$

Then, by Lemmata 2.1 and 2.2, for each $P \in \mathscr{G}$, $\tilde{w} \in \mathbb{W}$ and fixed $d \geq 0$, the unique solution to (12) is given by $\tilde{p} = T_d P(\tilde{w})$ and $\tilde{k} = K_{P+d}(\tilde{w})$. In particular, $T_0 = T$. The relation (13) shows that T_d inherits all properties derived above for T. In particular, T_d maps $\mathscr{G}$ into itself and $T_d(P) < \mathrm{id}_{\mathbb{W}}$ for all $P \in \mathscr{G}$.

2.2 Montonicity properties of T_d

We conjecture – and prove in the next subsection – that a fixed point of T_d, i.e., a function $P_d^* \in \mathscr{G}$ such that $P_d^* = T_d P_d^*$ together with the induced capital function $K_d^* = K_{P_d^*+d}$ define a ME of $\mathscr{E}_d$. In this regard, the last result from Lemma 2.2 implies $K_d^* \leq K_0$ with the latter defined by (5). This property permits to employ $\mathbb{W} =]0, w_{\max}]$ as the reduced state space and $\mathbb{X} = \mathbb{K} \times \Theta$ as the natural state space along *any* ME.

Our ultimate goal in this paper is to prove the existence of a BME which corresponds to a non-trivial fixed point $P_0^* > 0$ of T. Unfortunately, however, Lemma 2.2 already showed that the trivial solution $P = 0$ is always a fixed point of T, so a mere existence result will not help. Instead, we will explicitly construct fixed points as pointwise limits

[4] As K_P yields the solution K_0 defined by (5) for $P \equiv 0$, this notation is consistent with Section 1.4.

of function sequences. The method is similar to the one developed in Greenwood & Huffman (1995), see also Morand & Reffett (2003, 2007).

A key property for this construction to be successful is that T_d be monotonic which, by (13) is equivalent to monotonicity of T which we will consider first. A major obstacle to establish this property globally on $\mathscr{G}$ is that the methods from differential calculus including the implicit function theorem are not available for all functions in $\mathscr{G}$. To remedy this problem, we will temporarily restrict ourselves (respectively T) to the smaller set

$$\mathscr{G}' := \{P \in \mathscr{G} \mid P \text{ is } C^1\} \tag{14}$$

of continuously differentiable functions in $\mathscr{G}$. Observe that $\mathscr{G}'$ still contains the trivial solution $P \equiv 0$. The next result shows that T maps $\mathscr{G}'$ into itself.

Lemma 2.3
Under the hypotheses of Lemma 2.2, $P \in \mathscr{G}'$ implies $TP \in \mathscr{G}'$.

The following result now establishes the monotonicity of T on $\mathscr{G}'$ which will turn out to be sufficient to apply the construction principle below. In addition, we show that $K_\bullet$ is strictly decreasing on $\mathscr{G}'$ which resembles the usual crowding-out effect of assets.

Lemma 2.4
In addition to Assumptions 1–3, let (T1) and (U1) hold. Then, T is monotonically increasing on $\mathscr{G}'$, i.e., for any $P_0, P_1 \in \mathscr{G}'$, $P_1 \geq P_0$ implies $T(P_1) \geq T(P_0)$ and $P_1 > P_0$ implies $T(P_1) > T(P_0)$. Moreover, $K_\bullet$ is monotonically decreasing on $\mathscr{G}'$.

It follows directly from (13) that the operator T_d inherits all previous properties from T. In particular, T_d is monotonic on $\mathscr{G}'$ and maps this subclass into itself. In addition, the map $d \mapsto T_d$ is monotonic in the sense that $d_1 \geq d_0$ implies $T_{d_1}P \geq T_{d_0}P$ for all $P \in \mathscr{G}'$. For later reference, we state these properties formally in the next result.

Corollary 2.1
Under the hypotheses of Lemma 2.4, T_d satisfies the following monotonicity properties:

(i) For all $d \in \mathbb{R}_+$ and $P_0, P_1 \in \mathscr{G}'$: $P_1 \geq (>) P_0$ implies $T_d P_1 \geq (>) T_d P_0$.

(ii) For all $P \in \mathscr{G}'$ and $d_0, d_1 \in \mathbb{R}_+$: $d_1 \geq (>) d_0$ implies $T_{d_1}P \geq (>) T_{d_0}P$.

2.3 Constructing ME as fixed points of T_d

Let $d \geq 0$ be arbitrary but fixed. We are now in a position to construct ME of $\mathscr{E}_d$ as fixed points of T_d. For $m \in \mathbb{N}$, let T_d^m denote the m-fold composition of T_d with itself, i.e., $T_d^m = T_d \circ T_d^{m-1}$. As $T_d P = T(P + d) < \mathrm{id}_{\mathbb{W}}$ for all $P \in \mathscr{G}$ by Lemma 2.2, the identity map $\mathrm{id}_{\mathbb{W}} \in \mathscr{G}'$ defines a natural upper bound for any fixed point of T_d. Thus, define the sequence of functions $(P_d^m)_{m\geq 0}$ recursively by setting $P_d^0 \equiv P_0 := \mathrm{id}_{\mathbb{W}}$ and $P_d^m := T_d(P_d^{m-1}) = T_d^m P_0$. By Lemma 2.3, this sequence is well-defined and $P_d^m \in \mathscr{G}'$ for all $m \geq 0$. Further, $P_d^1 < P_d^0$ implies $P_d^{m+1} < P_d^m$ for all $m \geq 0$ by monotonicity of T_d, i.e., $(P_d^m)_{m\geq 0}$ is strictly decreasing. Thus, the pointwise limit

$$P_d^*(w) := \lim_{m\to\infty} P_d^m(w) = \lim_{m\to\infty} T_d^m P_0(w) \tag{15}$$

is well-defined for all $w \in \mathbb{W}$ as $(P_d^m(w))_{m\geq 0}$ is a strictly decreasing sequence bounded by zero. We show that the limiting function satisfies $P_d^* \in \mathscr{G}$. For each $m \geq 1$, $P_d^m \in \mathscr{G}$ implies that $w \mapsto P_d^m(w)$ and $w \mapsto w - P_d^m(w)$, $w \in \mathbb{W}$ are both increasing. Thus, for any $0 < w_1 < w_2 \leq w_{\max}$ the inequalities $P_d^m(w_1) \leq P_d^m(w_2)$ and $w_1 - P_d^m(w_1) \leq w_2 - P_d^m(w_2)$ being true for all $m \geq 1$ also hold in the limit and imply that P_d^* inherits the previous monotonicity properties. Using an argument developed and proved in Morand & Reffett (2003, p.1369), these properties already imply continuity of P_d^*. Thus, $P_d^* \in \mathscr{G}$. Note, however, that we can not be certain that $P_d^* \in \mathscr{G}'$.

The previous findings lead to the following main result.

Theorem 1
In addition to Assumptions 1–3, let (T1), (U1), and either (T2) or (U2) hold. Then, for each $d \geq 0$ the functions P_d^* defined in (15) and $K_d^* := K_{P_d^*+d}$ satisfy the following:

(i) P_d^* is a fixed point of T_d which satisfies $P_d^* > 0$ for $d > 0$ and either $P_0^* > 0$ or $P_0^* = 0$ if $d = 0$. Moreover, $d > d' \geq 0$ implies $P_d^* \geq P_{d'}^*$ and $K_d^* < K_{d'}^*$.

(ii) Both mappings P_d^* and K_d^* are continuous and increasing.

(iii) $K_d^E := K_d^* \circ W$ and $P_d^E := P_d^* \circ W$ is a ME of $\mathscr{E}_d$ on $\mathbb{X} = \mathbb{K} \times \Theta$.

For $d = 0$, the previous construction delivers a unique ME (K_0^*, P_0^*) of $\mathscr{E}$. Clearly, $P_0^* = 0$ implies $K_0^* = K_0$ defined by (5) which yields precisely the bubbleless equilibrium studied in Section 1.4. The main question of this paper, however, is when does $P_0^* > 0$ hold?

Before exploring this question in the next section, we present an alternative way to construct the ME from Theorem 1 for the benchmark economy $\mathscr{E}$. The proof of our main existence result will be based on this construction. The idea is to obtain the ME of $\mathscr{E}$ as the limit of ME of dividend economies $\mathscr{E}_d$ as d goes to zero. Formally, let $(d_n)_{n\geq 1}$ be a decreasing sequence of dividends such that $d_n \geq 0$ for all n and $\lim_{n\to\infty} d_n = 0$. By Theorem 1, for each $n \geq 1$ the functions $P_{d_n}^*$ defined by (15) and $K_{d_n}^* = K_{P_{d_n}^*+d_n}$ define a ME of $\mathscr{E}_{d_n}$. The following result shows that the sequence of ME constructed in this fashion indeed converges (pointwise) to the ME of $\mathscr{E}$ defined by Theorem 1.

Lemma 2.5
For any positive dividend sequence $(d_n)_{n\geq 1}$ converging monotonically to zero, the induced sequence of ME $(K_{d_n}^*, P_{d_n}^*)_{n\geq 1}$ from Theorem 1 converges pointwise to (K_0^*, P_0^*).

3 Existence of Bubbly Markov Equilibria

In this section we establish necessary and sufficient conditions under which the ME (K_0^*, P_0^*) constructed in Theorem 1 is bubbly, i.e., $P_0^* > 0$. Our main result stated as Theorem 2 below shows that this is the case whenever the bubbleless equilibrium derived in Section 1.4 is Pareto inefficient. As the proof requires that the (reduced) state space can be chosen compact along this equilibrium, the following sections replace our previous Assumption 3 by the stronger Assumption 4. In addition, the formal arguments in the proofs of Lemma 3.1 and Theorem 2 below assume that the shock space Θ is finite without explicit notice. These restrictions allow us to easily use the characterization of Pareto-inefficiency along with Proposition 4 from Barbie, Hagedorn & Kaul (2007). An extension to the case where Θ is an interval seems straightforward (but tedious) along the lines of Proposition 1 in Barbie & Kaul (2015). All other arguments and proofs in this section are formulated and hold for the general case where Θ is an interval.

In the following analysis, define $w_{\max}$ by (8) and $w_{\min}$ by (9). As a notational convention, a superscript * identifies functions associated with the ME constructed in Theorem 1.

3.1 Pareto optimality

Our concept of Pareto optimality corresponds to *Interim Pareto Optimality (IPO)* as defined and studied, e.g., in Demange & Laroque (2000) or *Conditional Pareto Optimality (CPO)* as in Chattopadhyay & Gottardi (1999). The following definition formalizes this concept for the class of economies $\mathscr{E}_d$ defined above for a fixed value $d \geq 0$.

Definition 4

(i) Given $x_0 = (k_0, \theta_0) \in \mathbb{X}$, a feasible allocation of $\mathscr{E}_d$ is an adapted stochastic process $a = \{k_{t+1}, c_t^y, c_t^o\}_{t\geq 0}$ with values in $\mathbb{R}_+^3$ which satisfies the resource constraint

$$k_{t+1} + c_t^y + c_t^o = f(k_t, \theta_t) + d$$

for all $t \geq 0$. The set of feasible allocations of $\mathscr{E}_d$ is denoted $\mathbb{A}_d(x_0)$.

(ii) Allocation $a \in \mathbb{A}_d(x_0)$ (Pareto) dominates allocation $\tilde{a} \in \mathbb{A}_d(x_0)$ if $c_0^o \geq \tilde{c}_0^o$ and

$$u_t := \mathbb{E}_t\left[u(c_t^y) + v(c_{t+1}^o)\right] \geq \mathbb{E}_t\left[u(\tilde{c}_t^y) + v(\tilde{c}_{t+1}^o)\right] =: \tilde{u}_t$$

for all $t \geq 0$ and for some $t \geq 0$ there exists a non-empty set $A \in \mathcal{F}_t$ such that $u_t(\omega) > \tilde{u}_t(\omega)$ for all $\omega \in A$.

(iii) Allocation $a \in \mathbb{A}_d(x_0)$ is called Pareto optimal or efficient if it is not dominated by any other allocation in $\mathbb{A}_d(x_0)$. Otherwise, it is called inefficient.

Our main result in Theorem 2 below establishes that the benchmark economy $\mathscr{E} = \mathscr{E}_0$ has a BME whenever the bubbleless equilibrium allocation is Pareto inefficient. To state this result formally, we introduce the concept of a *Markovian equilibrium allocation.*

3.2 Markovian equilibrum allocations (MEA)

For fixed $d \geq 0$, identify a ME of $\mathscr{E}_d$ with the mappings (K, P) on $\mathbb{W} =]0, w_{\max}]$ constructed as in the previous sections (here and in the sequel we drop the subscript d when convenient). We seek to derive the induced mappings which determine the consumption process along a ME. It will be convenient to define these mappings on the reduced state space $\mathbb{W}$ rather than $\mathbb{X}$ and to identify the state at time t by w_t. For this reason, we fix the realization of the initial shock $\theta_0 \in \Theta$[5] and define the consumption mappings associated with a ME (K, P) as

$$\begin{array}{llll} C^y : \mathbb{W} \longrightarrow \mathbb{R}_{++}, & C^y(w) & := & w - K(w) - P(w) \\ C^o : \mathbb{W} \times \Theta \longrightarrow \mathbb{R}_{++}, & C^o(w, \theta) & := & P(W(K(w), \theta)) + d + K(w)R(K(w), \theta). \end{array} \tag{16}$$

We call the triple $A = (K, C^y, C^o)$ a *Markovian Equilibrium Allocation* (MEA). The *pricing kernel* associated with A is defined as the map $m_A : \mathbb{W} \times \Theta \longrightarrow \mathbb{R}_{++}$,

$$m_A(w, \theta) := \frac{v'(C^o(w, \theta))}{u'(C^y(w))}. \tag{17}$$

For each $w_0 \in \mathbb{W}$, a MEA determines a unique feasible allocation $a^E(w_0) \in \mathbb{A}_d(x_0)$ where $k_{t+1} = K(w_t)$, $c_t^y = C^y(w_t)$, $c_{t+1}^o = C^o(w_t, \theta_{t+1}) = C^o(w_t, w_{t+1}/W(K(w_t), 1)$ for $t \geq 0$ while old-age consumption c_0^o in $t = 0$ follows from the aggregate resource constraint. Consequently, we adopt the following notions of efficiency/inefficiency for MEA.

Definition 5
A MEA $A = (K, C^y, C^o)$ is called

(i) efficient/inefficient at $w_0 \in \mathbb{W}$ if $a^E(w_0)$ is efficient/inefficient.

(ii) efficient/inefficient on $\overline{\mathbb{W}} \subset \mathbb{W}$ if A is efficient/inefficient at all $w_0 \in \overline{\mathbb{W}}$.

(iii) efficient/inefficient if it is efficient/inefficient at each $w_0 \in \mathbb{W}$.[6]

The previous formulation permits consumption and capital along the ME to be expressed as functions of the (reduced) state process $\{w_t\}_{t \geq 0}$. Given $w_0 \in \mathbb{W}$, the statistical evolution of this process is determined by a time-invariant transition probability Q (see Appendix B for details). Therefore, the lifetime utility u_t of generation t from Definition 4 (ii) also depends exclusively on the state w_t. Combining results from Barbie, Hagedorn & Kaul (2007) and Barbie & Kaul (2015), these properties will allow us to characterize

[5]This restriction is necessary because initial old-age consumption c_0^o can, in general, not be written as a function of w_0 but requires knowledge of the full initial state x_0. If θ_0 is fixed, there is a one-to one correspondence between w_0 and the initial state x_0 and the process $\{x_t\}_{t \geq 0}$ can fully be recovered from $\{w_t\}_{t \geq 0}$ as $k_t = K(w_{t-1})$ and $\theta_t = w_t/W(k_t, 1)$ for $t \geq 1$.

[6]Under the additional restrictions from Lemma 3.1 (ii) below, the efficiency properties of A become to some extent independent of the initial state w_0.

the (in-)efficiency of MEA in terms of mappings defined on a one-dimensional state space which greatly simplifies this characterization. To obtain these results, the following additional restrictions on MEA will be employed.

Definition 6
Let $A = (K, C^y, C^0)$ be a MEA defined as above.

(i) We call A continuous *if the mappings K, C^y, and C^o are all continuous.*

(ii) We call a subset of the form $\overline{\mathbb{W}} = [\underline{w}, w_{\max}] \subset \mathbb{W}$ a stable set *and $\underline{w} > 0$ a* lower bound *(of A) if $w \in \overline{\mathbb{W}}$ implies $W(K(w), \theta) \in \overline{\mathbb{W}}$ for all $\theta \in \Theta$.*

(iii) We call A bounded*, if for each $w_0 \in \mathbb{W}$ there is some stable set $\overline{\mathbb{W}}$ containing w_0.*

For each $d \geq 0$, denote by $A_d^* = (K_d^*, C_d^{y,*}, C_d^{o*})$ the MEA associated with the ME (K_d^*, P_d^*) from Theorem 1. Further, let $A_0 = (K_0, C_0^y, C_0^o)$ be the MEA associated with the bubbleless ME of $\mathscr{E}$ derived in Section 1.4. That is, K_0 is defined by (5), $C_0^y(w) := w - K_0(w)$, and $C_0^o(w, \theta) := K_0(w)R(K_0(w), \theta)$ for all $w \in \mathbb{W}$ and $\theta \in \Theta$. Note that A_0^* coincides with A_0 if and only if (K_0^*, P_0^*) is bubbleless, i.e., $P_0^* = 0$. This observation will play a key role in the next section. Also observe that A_0 and each A_d^* are continuous by the results from Section 1.4 and Theorem 1 (ii) and that A_0 is bounded under the additional restriction from Assumption 4.

3.3 A general existence theorem

We are now in a position to state our main existence result in the following theorem.

Theorem 2
In addition to Assumptions 1, 2, and 4, let (U1), (T1), and either (U2) or (T2) hold. If A_0 is inefficient, then (K_0^*, P_0^*) defines a BME of $\mathscr{E}$, i.e., $P_0^* > 0$.

The intuition behind the proof of Theorem 2 is straightforward. Consider a monotonic sequence of strictly positive dividend payments $(d_n)_{n \geq 1}$ which converges to zero. For each $n \geq 1$, construct the ME $(K_{d_n}^*, P_{d_n}^*)$ of $\mathscr{E}_{d_n}$ as in Theorem 1 and denote by $A_{d_n}^*$ the induced MEA defined as above. It is well-known that each $A_{d_n}^*$, being an equilibrium allocation of an economy with a dividend-paying asset, is efficient. Intuitively, one would expect that this efficiency also holds in the limit such that the sequence $(A_{d_n}^*)_{n \geq 1}$ can not converge to A_0 if A_0 is inefficient. Thus, $A_0 \neq A_0^*$ which is only possible if $P_0^* > 0$, i.e., (K_0^*, P_0^*) is bubbly.[7]

[7] The same argument is used in Barbie & Kaul (2015), going back to the basic idea in Aiyagari & Peled (1991), for the case of an exchange economy, where instead of the monotonicity methods applied here Schauder's fixed point theorem is used. Since in our framework in addition the capital stock adjusts as an endogenous variable, the analysis becomes more complicated than under pure exchange.

We preface the proof of Theorem 2 by the following three lemmata. The first result is a sort of unit root condition, which is used in OLG models with finitely many states to characterize the Pareto optimality of stationary competitive equilibria. The proof is an adaption of the results from Barbie, Hagedorn & Kaul (2007) and Barbie & Kaul (2015) and is relegated to Appendix B. Note the similarity of (18) to the conditions for inefficiency in Demange & Laroque (2000) or Magill & Quinzii (2003).

Lemma 3.1
Let $A = (K, C^y, C^o)$ be a MEA which is continuous and bounded.

(i) If A is inefficient, there is an upper-semi-continuous function $\eta : \mathbb{W} \longrightarrow]0, 1]$ such that

$$\mathbb{E}_\nu [\eta(W(K(w), \cdot))\, m_A(w, \cdot)] > \eta(w) \text{ for all } w \in \mathbb{W}. \tag{18}$$

(ii) If m_A in (17) is increasing, then η in (i) can be chosen continuous. Moreover, if A is inefficient at some $w_0 \in \mathbb{W}$, it is also inefficient for all $w_0' \geq w_0$.

Let $m_0 := m_{A_0}$ be the pricing kernel associated with the bubbleless allocation A_0. Our next result ensures that η in (18) can be chosen continuous whenever A_0 is inefficient.

Lemma 3.2
If Assumptions 1–3, (T1), (U1), and either (T2) or (U2) hold, then m_0 is increasing.

Finally, we have the following sufficient condition for inefficiency. This condition also appears as part of Theorem 1 of Barbie & Kaul (2015) and as Theorem 1 in Demange & Laroque (2000). The proof we give here is similar to the ones given in these papers.

Lemma 3.3
Let $A = (K, C^y, C^o)$ be continuous and $\overline{\mathbb{W}}$ be a stable set of A. If a continuous function $\eta : \overline{\mathbb{W}} \longrightarrow]0, 1]$ satisfies (18) for all $w \in \overline{\mathbb{W}}$, then A is inefficient on $\overline{\mathbb{W}}$.

We are now in a position to prove Theorem 2 in five steps.

Step 1: Let $w_0 \in \mathbb{W}$ be arbitrary and $\overline{\mathbb{W}} = \overline{\mathbb{W}}_{A_0} = [\underline{w}, w_{\max}]$ be a stable set of A_0 containing w_0 such that $W(K_0(\underline{w}), \theta_{\min})/\underline{w} > 1$. Assumption 4 ensures that such a set exists. By hypothesis, A_0 is inefficient at w_0. Thus, invoking Lemmata 3.1 and 3.2, there exists a continuous function $\eta : \mathbb{W} \to]0, 1]$ such that for all $w \in \overline{\mathbb{W}}$:

$$\mathbb{E}_\nu \big[\eta(W(K_0(w), \cdot))m_0(w, \cdot)\big] > \eta(w). \tag{19}$$

Step 2: Define the sequence $(d_n)_{n\geq1}$ as $d_n := \bar{d}/n$ for $n \geq 1$ with $\bar{d} > 0$ specified below. For each $n \geq 1$, let $(K^*_{d_n}, P^*_{d_n})$ be the ME of $\mathscr{E}_{d_n}$ from Theorem 1 and define the induced MEA $A^*_{d_n} = (K^*_{d_n}, C^{y*}_{d_n}, C^{o*}_{d_n})$ as in Section 3.2. By Lemma 2.5, the sequence $(K^*_{d_n}, P^*_{d_n})_{n\geq1}$ converges pointwise to the ME (K^*_0, P^*_0) of $\mathscr{E}$ which satisfies either $P^*_0 = 0$ or $P^*_0 > 0$. We will show that the first case is impossible under the hypotheses of the theorem. Thus, with the aim of obtaining a contradiction, the remainder assumes $P_0^* = 0$ which implies $K_0^* = K_0$. Then, the sequence $(A_{d_n}^*)_{n\geq 1}$ converges pointwise to $A_0 = (K_0, C_0^y, C_0^o)$ defined above. Further, the sequence $(m_n)_{n\geq 1}$ of pricing kernels $m_n := m_{A_{d_n}^*}$ associated with $A_{d_n}^*$ defined in (17) converges pointwise to $m_0 = m_{A_0}$.

Step 3: We choose $\bar{d} > 0$ such that $\overline{\mathbb{W}} = [\underline{w}, w_{\max}]$ is stable for each $A_{d_n}^*$. Since $(K_{d_n}^*)_{n\geq 1}$ is increasing by Theorem 1 (i), it suffices to specify $\bar{d}$ such that $\overline{\mathbb{W}}$ is stable for $A_{d_1}^*$. As $\delta := W(K_0(\underline{w}), \theta_{\min})/\underline{w} > 1$ and $K_{d_1}^* = K_{\bar{d}}^*$ converges pointwise to K_0 for $\bar{d} \searrow 0$ due to Lemma 2.5, choosing $\bar{d} > 0$ small ensures $W(K_{\bar{d}}^*(\underline{w}), \theta_{\min})/\underline{w} > 1$. Then, $w \geq \underline{w}$ implies $W(K_{d_n}^*(w), \theta) \geq W(K_{d_1}^*(w), \theta_{\min}) \geq W(K_{d_1}^*(\underline{w}), \theta_{\min}) > \underline{w}$, i.e., $\overline{\mathbb{W}}$ is stable for $A_{d_n}^*$.

Step 4: Standard arguments imply that each $A_{d_n}^*$ is efficient on $\overline{\mathbb{W}}$. To see this, define for $n \geq 1$ the continuous function $R_n^*(w, \theta) := (P_{d_n}^*(W(K_{d_n}^*(w), \theta)) + d_n)/P_{d_n}^*(w)$ which satisfies $\mathbb{E}_\nu\,[m_n(w, \cdot)R_n^*(w, \cdot)] = 1$ for all $w \in \overline{\mathbb{W}}$. Thus, R_n^* is *a return* in the sense of Barbie, Hagedorn & Kaul (2007), cf. their equation (5). For all $T > 0$ and $w_0 \in \overline{\mathbb{W}}$, monotonicity of $P_{d_n}^*$ implies $\Pi_{t=1}^T R_n^*(w_{t-1}, \theta_t) \geq P_{d_n}^*(w_T)/P_{d_n}^*(w_0) \geq P_{d_n}^*(\underline{w})/P_{d_n}^*(w_{\max}) =: M$ for any realization of shocks $\theta_1, \ldots, \theta_T$ where $w_t = W(K_{d_n}^*(w_{t-1}), \theta_t)$. Note that M is independent of T and the shocks. Using Proposition 4(a) in Barbie, Hagedorn & Kaul (2007), this implies that $A_{d_n}^*$ is interim Pareto efficient on $\overline{\mathbb{W}}$.[8]

Step 5: Combining the previous result with Lemma 3.3 shows that for each $n \geq 1$ there exists some $w_n \in \overline{\mathbb{W}}$ such that

$$\mathbb{E}_\nu\big[\eta(W(K_{d_n}^*(w_n), \cdot))m_n(w_n, \cdot)\big] \leq \eta(w_n). \tag{20}$$

Since $\overline{\mathbb{W}}$ is compact, the sequence $(w_n)_{n\geq 1}$ contains a subsequence converging to some $w^* \in \mathbb{W}$. Denote this sequence again by $(w_n)_{n\geq 1}$. Clearly, $\lim_{n\to\infty} \eta(w_n) = \eta(w^*)$ by continuity of η. We would like to show that for all $\theta \in \Theta$

$$\lim_{n\to\infty} \eta(W(K_{d_n}^*(w_n), \theta))m_n(w_n, \theta) = \eta(W(K_0(w^*), \theta))m_0(w^*, \theta). \tag{21}$$

Since all functions in (21) are continuous, it suffices to show that $\lim_{n\to\infty} P_{d_n}^*(w_n) = 0$ and $\lim_{n\to\infty} K_{d_n}^*(w_n) = K_0(w^*)$. We have that $\lim_{n\to\infty} \sup\{P_{d_n}^*(w)\,|\,w \in \overline{\mathbb{W}}\} = 0$ by Theorem A in Buchanan & Hildebrandt (1908)[9], which immediately gives the result for $P_{d_n}^*$. Also by Theorem A in Buchanan & Hildebrandt (1908), $(K_{d_n}^*)_{n\geq 1}$ being a sequence of strictly monotonic functions converges uniformly to K_0 on the compact interval $\overline{\mathbb{W}}$. Combined with continuity of K_0, for any $\delta > 0$ there exists n_0 such that $n > n_0$ implies

$$|K_{d_n}^*(w_n) - K_0(w^*)| \leq |K_{d_n}^*(w_n) - K_0(w_n)| + |K_0(w_n) - K_0(w^*)| < \delta.$$

[8]The result that economies with a dividend paying asset have efficient equilibria is well-known and can also by proved by defining state contingent claims prices and showing that the value of the aggregate endowment is finite (due to the presence of dividends). Efficiency of the equilibrium allocation then follows along the lines of the standard proof of the first welfare theorem.

[9]Their result states the simple, but in our analysis very useful fact that if a sequence of monotonic real-valued functions f_n defined on the interval $[a, b]$ with $a < b$ converges pointwise to a continuous function f on $[a, b]$, then f is also monotonic and convergence is uniform.

This proves (21). Further, η is bounded as a continuous function on the compact set $\overline{\mathbb{W}}$ while $0 < m_n(w_n, \theta) < v'(K_{\bar{d}}(\underline{w})R(K_{\bar{d}}(\underline{w}), \theta_{\min}))/u'(w_{\max})$ for each $\theta \in \Theta$. Thus, by the Lebesgue-dominated convergence theorem, (20) and (21) imply

$$\mathbb{E}_\nu\big[\eta(W(K_0(w^*), \cdot))m_0(w^*, \cdot)\big] \leq \eta(w^*). \tag{22}$$

But this contradicts (19) and proves the claim that $P_0^* > 0$. ∎

The previous construction also suggests that the limiting MEA A_0^* associated with the BME (K_0^*, P_0^*) is efficient. Clearly, if A_0^* is bounded, this follows immediately from the same arguments used in Step 4. Unfortunately, however, boundedness of A_0^* is not guaranteed even if the bubbleless equilibrium satisfies Assumption 4.

Under the hypotheses of Theorem 2, suppose the shock process is non-degenerate, i.e., $\theta_{\min} < \theta_{\max}$ or, equivalently, $w_{\min} < w_{\max}$ defined by (8) and (9). Then, for any initial value $w_0 \in \mathbb{W}$, the dynamics (7) takes values in the ergodic set $[w_{\min}, w_{\max}]$ after finitely many periods with positive probability. In this case, global inefficiency of A_0 is equivalent to inefficiency on the ergodic set which, by Lemma 3.1 (ii) and 3.2 is equivalent to A_0 being inefficient at $w_{\min}$. Thus, we obtain the following existence result as a corollary to Theorem 2.

Corollary 3.1
In addition to Assumptions 1, 2, and 4, let (U1), (T1), and either (U2) or (T2) hold. If $w_{\min} < w_{\max}$ and A_0 is inefficient at $w_{\min}$, then (K_0^*, P_0^*) is a BME of $\mathscr{E}$, i.e., $P_0^* > 0$.

3.4 Conditions for inefficiency of A_0

In this section we provide necessary and sufficient conditions for A_0 to be inefficient as required in Theorem 1 which are simple and easy to verify. As in the previous section, we impose the stronger Assumption 4 and define $w_{\max}$ by (8) and $w_{\min}$ by (9).

Define the bubbleless MEA $A_0 = (K_0, C_0^y, C_0^o)$ as before. The pricing kernel $m_0 = m_{A_0}$ defined in (17) induces a map $M : \mathbb{W} \longrightarrow \mathbb{R}_{++}$,

$$M(w) := \mathbb{E}_\nu\big[m_0(w, \cdot)\big]. \tag{23}$$

Economically, the value $1/M(w)$ can be interpreted as the riskless return in state $w \in \mathbb{W}$. Using (5) and the definition (17) of m_0, M can equivalently be written as

$$M(w) = \frac{\mathbb{E}_\nu[v'(C_0^o(w, \cdot))]}{\mathbb{E}_\nu[R(K_0(w), \cdot)v'(C_0^o(w, \cdot))]}, \quad w \in \mathbb{W}. \tag{24}$$

The representation in (24) reveals directly that M is continuously differentiable and satisfies $0 \leq M(w) \leq 1/R(K_0(w); \theta_{\min})$ for all w. The latter implies $\lim_{w \searrow 0} M(w) = 0$.

Our first result states a simple sufficient condition under which A_0 is inefficient. Note that the additional restrictions (T2) or (U2) are not required here.

Lemma 3.4
Let Assumptions 1, 2, 4, and (T1) and (U1) hold and define M as in (23). If $M(w) > 1$ for all $w \in [w_{\min}, w_{\max}]$, then A_0 is inefficient.

Proof: We construct a continuous function $\eta :]0, w_{\max}] \longrightarrow \mathbb{R}_{++}$ which satisfies (19) for all $w \in \mathbb{W}$. By Lemma 3.3, this implies inefficiency of A_0 on any stable set $[\underline{w}, w_{\max}]$ which implies inefficiency on $\mathbb{W}$.

Defining W_0^E as in (7), note that $W_0^E(\cdot; \theta_{\min})$ is strictly increasing and, therefore, invertible on its range. Denote the inverse by Λ. By continuity of M, there exists $\delta > 0$ such that $M(w) > 1$ for all $w \in [w_{\min} - \delta, w_{\max}]$. Construct a sequence $(w_n)_{n \geq 0}$ by setting $w_0 := w_{\min} - \delta$ and $w_n := \Lambda(w_{n-1}) = \Lambda^n(w_0)$ for $n \geq 1$. Note that $(w_n)_{n \geq 0}$ is strictly decreasing and, due to Assumption 4, converges to zero.

Now construct η as follows. For $w \in [w_0, w_{\max}]$, let $\eta(w) \equiv 1$. Then,

$$\mathbb{E}_\nu\big[\eta(W_0^E(w, \cdot))m_0(w, \cdot)\big] = M(w) > 1 = \eta(w)$$

for all $w \in [w_0, w_{\max}]$. Second, for $w \in [w_1, w_0[$ let $\eta(w) := M(w)/M(w_0)$. Then,

$$\mathbb{E}_\nu\big[\eta(W_0^E(w, \cdot))m_0(w, \cdot)\big] = M(w) > M(w)/M(w_0) = \eta(w)$$

for all $w \in [w_1, w_0[$. Now proceed inductively for $n \geq 1$ by defining for $w \in [w_n, w_{n-1}[$

$$\eta(w) := \mathbb{E}_\nu\big[\eta(W_0^E(w, \cdot))m_0(w, \cdot)\big]/M(w_0).$$

By construction, η is a continuous function which satisfies (19). Since $[w_0, w_{\max}] \cup (\cup_{n \geq 1}[w_n, w_{n-1}[) =]0, w_{\max}]$ the previous construction covers the entire interval $\mathbb{W}$. ■

A partial converse to Lemma 3.4 is the following result.

Lemma 3.5
Let Assumptions 1, 2, 4, and (T1) and (U1) hold. If A_0 is inefficient, then $M(w) > 1$ for at least one $w \in [w_{\min}, w_{\max}]$.

Proof: By contradiction, suppose A_0 is inefficient but $M(w) \leq 1$ for all $w \in [w_{\min}, w_{\max}]$. By Lemma 3.1, there is an upper-semi-continuous function $\eta : \mathbb{W} \longrightarrow \mathbb{R}_{++}$ such that

$$\mathbb{E}_\nu\big[\eta(W_0^E(w, \cdot))m_0(w, \cdot)\big] > \eta(w).$$

for all $w \in [w_{\min}, w_{\max}]$. By Theorem 2.43 in Aliprantis & Border (2007, p.44), η attains a maximum on any compact set and the set of maximizers is compact. Letting $w^* \in [w_{\min}, w_{\max}]$ be a value for which $\eta(w^*) = \eta_{\max} := \max\{\eta(w) \,|\, w \in [w_{\min}, w_{\max}]\}$,

$$\mathbb{E}_\nu\big[\eta(W_0^E(w^*, \cdot))m_0(w^*, \cdot)\big] \leq \eta_{\max}\mathbb{E}_\nu\big[m_0(w^*, \cdot)\big] = \eta_{\max}M(w^*) \leq \eta_{\max} = \eta(w^*)$$

which is a contradiction. ■

The previous conditions take an even simpler form under the additional restrictions (T2) or (U2). In this case, monotonicity of m_0 due to Lemma 3.2 implies that M is strictly increasing. Combining Lemmata 3.4 and 3.5 then leads to the following main result.

Theorem 3
In addition to Assumptions 1, 2, and 4, let (T1), (U1), and either (T2) or (U2) hold.

(i) If $M(w^{\min}) > 1$, then A_0 is inefficient.

(ii) If A_0 is inefficient, then $M(w^{\max}) > 1$.

In the deterministic case where $w_{\min} = w_{\max}$, the two conditions from Theorem 3 reduce to $M(w_{\min}) > 1$ which is equivalent to a capital return $R < 1$ at the bubbleless steady state. This is precisely the condition in Tirole (1985) which is sufficient *and* necessary in the deterministic case. In the present stochastic case, the condition $M > 1$ requires an 'average' capital return less than unity on the ergodic set $[w_{\min}, w_{\max}]$.

3.5 An example economy

The following example illustrates the construction of ME of $\mathscr{E}$ developed in Section 2 and the previous conditions under which the ME is bubbly. We also demonstrate that the condition $M(w_{\min}) > 1$ from Theorem 3 is not necessary for a BME to exist.

Suppose $f(k) = k^\alpha$, $0 < \alpha < 1$, $u(c) = \log(c)$, and $v(c) = \beta u(c)$, $\beta > 0$. This parametrization is widely studied in the literature, cf. Michel & Wigniolle (2003) or Demange & Laroque (2000). Rangazas & Russell (2005) provide a detailed discussion of the (dynamic) efficiency properties of the bubbleless equilibrium allocation.

One verifies directly that Assumptions 1 and 2 and the additional restrictions (T1), (U1), and (U2) hold. Moreover, the mapping K_0 associated with the bubbleless ME of $\mathscr{E}$ defined by (5) computes $K_0(w) = \frac{\beta}{1+\beta} w$ such that W_0^E defined in (7) takes the form

$$W_0^E(w, \theta) = \theta(1-\alpha)\left(\frac{\beta}{1+\beta} w\right)^\alpha. \tag{25}$$

Direct computations reveal that $W_0^E(\cdot, \theta_{\max})$ has a unique non-trivial fixed point given by $w_{\max} = [(1-\alpha)\theta_{\max}(\beta/(1+\beta))^\alpha]^{\frac{1}{1-\alpha}}$ which is stable. Further, $W_0^E(\cdot, \theta_{\min})$ also has a unique fixed point $w_{\min} = [(1-\alpha)\theta_{\min}(\beta/(1+\beta))^\alpha]^{\frac{1}{1-\alpha}}$ and Assumption 4 is satisfied. For later reference, let $k_{\max} := K_0(w_{\max})$ denote the maximum capital stock and $R_{\max} := R(k_{\max}, \theta_{\max})$ the associated maximum capital return. These values compute explicitly as $k_{\max} = [\frac{\beta}{1+\beta}(1-\alpha)\theta_{\max}]^{\frac{1}{1-\alpha}}$ and $R_{\max} = \frac{1+\beta}{\beta}\frac{\alpha}{1-\alpha}$.

Applying the construction principle from Section 2.3, let $P_0 = \mathrm{id}_{\mathbb{W}}$ and consider the sequence $(P_n)_{n\geq 0}$ defined as $P_n = T(P_{n-1})$, $n \geq 1$. As $P(w) = \delta w$ implies $TP(w) = [R_{\max} + \delta\frac{1+\beta}{\beta}]^{-1} P(w)$ for $w \in \mathbb{W}$, the operator T maps linear functions onto linear functions. Thus, each P_n is linear and can be computed explicitly as

$$P_n(w) = \frac{w}{R_{\max}^n + \frac{1+\beta}{\beta}\sum_{m=0}^{n-1} R_{\max}^m}, n \geq 0.$$

For each $w \in \mathbb{W}$, the limit P_0^* defined in (15) is given by

$$P_0^*(w) = \begin{cases} (\frac{\beta}{1+\beta} - \frac{\alpha}{1-\alpha})w & \text{if } R_{\max} < 1 \\ 0 & \text{otherwise.} \end{cases} \tag{26}$$

Thus, in this example, the ME constructed is bubbly, if and only if $R_{\max} < 1$ which is equivalent to A_0 being Pareto inefficient. To relate this result to the condition in Theorem 3 (ii), consider the function M defined in (24) which can be computed as

$$M(w) = \mathbb{E}_\nu[(R(K_0(w), \cdot))^{-1}]. \tag{27}$$

One verifies by direct computations that in this case, $M(w_{\max}) = \frac{1}{R_{\max}}\mathbb{E}_\nu[\theta_{\max}/(\cdot)]$ and $M(w_{\min}) = \frac{1}{R_{\max}}\mathbb{E}_\nu[\theta_{\min}/(\cdot)]$. As $\mathbb{E}_\nu[\theta_{\max}/(\cdot)] > 1$, $R_{\max} < 1$ implies $M(w_{\max}) > 1$. On the other hand, one can easily choose a distribution ν such that $\mathbb{E}_\nu[\theta_{\min}/(\cdot)] < R_{\max} < 1$. In this case, the fixed point in (26) satisfies $P_0^* > 0$ and $\mathscr{E}$ has a BME even though $M(w_{\min}) < 1$.

In the previous example, the ME defined by (15) is bubbly, if and only if $R_{\max} < 1$. We remark that the same condition can be used to ensure existence of a BME in the more general case where (U1) holds and both u and v display constant relative risk aversion (of the same degree) while f satisfies the restriction (T1') $E_f + E_{f'} \leq 1$ which is slightly stronger than (T1) but also holds in the Cobb-Douglas case. Again, Assumptions 1-4 are all satisfied for this example. The condition $R(k_{\max}; \theta_{\max}) < 1$ now ensures existence of a linear function $\underline{P}(w) = \delta w$, $w \in \mathbb{W}$, $0 < \delta < 1$ which is a lower bound for T in the sense that $T\underline{P} > \underline{P}$.[10] By the monotonicity properties of T, the fixed point in (15) satisfies $P_0^* > \underline{P} > 0$ and, therefore, defines a BME. While sufficient, it is not clear whether $R_{\max} < 1$ is also necessary for a BME to exist in this more general case and whether $R_{\max} < 1$ is sufficient in more general cases. We suspect that, in general, a characterization of inefficiency simpler than the one in Lemma 3.1 is not available.

3.6 Dynamics along a BME

Suppose the ME (K_0^*, P_0^*) of $\mathscr{E}$ constructed in Theorem 1 is bubbly, i.e., $P_0^* > 0$. We seek to deduce several qualitative properties of the equilibrium dynamics along a BME.

Given the initial state $x_0 = (k_0, \theta_0) \in \mathbb{X}$, all equilibrium variables can be expressed as continuous functions of the equilibrium wage process $\{w_t\}_{t\geq 0}$ which evolves as

$$w_{t+1} = W^E(w_t, \theta_{t+1}) := W(K_0^*(w_t), \theta_{t+1}). \tag{28}$$

It will again be convenient to study (28) rather than the mathematically equivalent capital dynamics generated by $K^E = K_0^* \circ W$.

[10]The proof of this result is straightforward but rather tedious and, therefore, omitted. It is available from the authors upon request.

As $P_0^* > 0$ implies $K_0^* < K_0$, a first observation is that $W^E < W_0^E$ where the latter is defined in (7). Thus, the sequence generated by (28) is bounded by the wage process (7) along the bubbleless equilibrium under any path of the shock process $\{\theta_t\}_{t\geq 0}$.

A second observation that follows from the Euler equations (3a,b) is that in each period the return on the bubbly asset must (weakly) exceed the capital return (1b) in at least one future state. Thus, for each $w \in \mathbb{W}$ there exists some $\theta' \in \Theta$ such that

$$\frac{P_0^*(W^E(w,\theta'))}{P_0^*(w)} \geq R(K_0^*(w),\theta'). \tag{29}$$

As $\lim_{w\to 0} R(K_0^*(w),\theta') = \infty$ for all $\theta' \in \Theta$ and the left side in (29) is increasing in the shock, there exists a lower bound $\underline{w}' > 0$ such that $R(K_0^*(w),\theta_{\min}) > 1$ and $P_0^*(W^E(w,\theta_{\max})) > P_0^*(w)$ for all $w \leq \underline{w}'$. Thus, by monotonicity of P_0^*

$$W^E(w,\theta_{\max}) > w \tag{30}$$

for all $w \leq \underline{w}'$. As $W^E(w_{\max},\theta_{\max}) < W_0^E(w_{\max},\theta_{\max}) = w_{\max}$, (30) also shows that $W^E(\cdot,\theta_{\max})$ has at least one stable fixed point which lies in the interval $]\underline{w}', w_{\max}[$. In fact, since $\theta_{\max}$ belongs to the support of ν, (30) and continuity of W^E imply that for each $w \leq \underline{w}'$ there exists a measurable set $\Theta_w \subset \Theta$ of positive measure $\nu(\Theta_w) > 0$ such that $W^E(w,\theta) > w$ holds for all $\theta \in \Theta_w$. Thus, defining $\underline{p}^* := P_0^*(\underline{w}')$, one observes that the bubbly asset price process $\{p_t\}_{t\geq 0}$ along the BME is persistent in the sense that whenever $p_t < \underline{p}^*$ there is a positive probability that $p_{t+n} > \underline{p}^*$ for some finite $n \geq 1$.

An open question is whether this last result can be strengthened in the sense that $p_t < \underline{p}^*$ implies $p_{t+n} > \underline{p}^*$ for some finite $n \geq 1$ with probability one. Essentially, this holds when the wage dynamics (28) admits a uniform lower bound $\underline{w}' > 0$ such that $W^E(w,\theta) > w$ for all $\theta \in \Theta$ whenever $w \leq \underline{w}'$. The example from Section 3.5 satisfies this condition. If such a lower bound exists, the bubble price processes and in fact all equilibrium variables remain bounded away from zero with probability one. Clearly, Assumption 4 is a necessary precondition for this to hold, but is it sufficient? This question becomes particularly relevant for studying the existence of stationary distributions associated with the state process defined by (28) which we leave for future research.

4 Conclusions

This paper developed a general approach to construct potentially bubbly Markov equilibria for a general class of frictionless OLG economies with stochastic production. Our main result shows that a BME exists whenever the bubbleless equilibrium is inefficient. This type of inefficiency can be the result of an *overaccumulation of capital* but also due to *inefficient risk sharing* between consumers. The deterministic result of Tirole (1985) therefore constitutes a special case of our existence theorem.

To focus on this existence result and keep the technical part bearable, we deliberately limited the underlying class of economies to a setup with i.i.d. TFP shocks and additive consumer utility. We believe that these restrictions are inessential and easy to dispense with at a cost of a more complex structure and notation. Potential extensions of the previous framework include non-additive consumer utility, non-multiplicative and correlated production shocks, and non-classical production technologies. These extensions were employed, e.g., in Wang (1994), Morand & Reffett (2007), McGovern, Morand & Reffett (2013), or Hillebrand (2014) to study the existence and properties of bubbleless ME. Since all these papers rely on methods similar to those employed in this paper, we believe that the previous construction of a BME should be amendable to these extensions. This constitutes a first major objective of future research.

In addition, several issues remain to be studied even within the framework of this paper. For instance, an open question is if the bubbly equilibrium is always efficient and, related to that, whether it constitutes a Pareto improvement relative to the bubbleless equilibrium. The characterization of Pareto optimality developed in Section 3 should be key to answer these questions. Another avenue of future research is whether the state dynamics along the BME converge to a stationary distribution on the endogenous state space. Since our equilibrium mappings are all monotonic, we view the recent results of Kamihigashi & Stachurski (2014) as tailor-made for studying the existence, uniqueness, and stability of stationary distributions along a BME.

A Mathematical Appendix

A.1 Proof of Lemma 2.1

(i) Let $P \in \mathscr{G}$ be given and $w \in \mathbb{W}$ be arbitrary but fixed. For each $k \in \mathbb{K} =]0, k_{\max}]$ and $\theta \in \Theta$, set $c(k,\theta) := P(W(k,\theta)) + kR(k,\theta)$ which is a strictly increasing function due to monotonicity of P and (T1). For $k \in \mathbb{K}$, define the functions

$$\tilde{P}(k) := \frac{\mathbb{E}_\nu\big[P(W(k,\cdot))v'\big(c(k,\cdot)\big)\big]}{\mathbb{E}_\nu\big[R(k,\cdot)v'\big(c(k,\cdot)\big)\big]} \tag{A.1}$$

and

$$S(k) := k + \tilde{P}(k) = \frac{\mathbb{E}_\nu\big[c(k,\cdot)v'\big(c(k,\cdot)\big)\big]}{\mathbb{E}_\nu\big[R(k,\cdot)v'\big(c(k,\cdot)\big)\big]} =: \frac{\tilde{N}(k)}{D(k)}. \tag{A.2}$$

Since P is continuous, so are the mappings $\tilde{P}$, $\tilde{N}$, D, and S. Observe that $\tilde{N}$ in (A.2) is weakly increasing due to (U1) and monotonicity of $c(\cdot,\theta)$ while D is strictly decreasing which implies that S is strictly increasing. Furthermore, by the boundary conditions imposed in Assumptions 1 and 2

$$\lim_{k\to 0} D(k) = \infty \tag{A.3}$$

which, together with the monotonicity of $\tilde{N}$ implies

$$0 \leq \lim_{k \to 0} \tilde{P}(k) \leq \lim_{k \to 0} S(k) = \lim_{k \to 0} \frac{\tilde{N}(k)}{D(k)} = 0. \tag{A.4}$$

For $k \in \mathbb{K}$, define

$$G(k; w) := u'(w - S(k)) - D(k). \tag{A.5}$$

Then, the desired solution $\tilde{k}$ solves $G(\tilde{k}; w) = 0$. Observe that $G(\cdot; w)$ is a strictly increasing function which follows from the monotonicity of S and D and u'. Thus, any zero is necessarily unique. Also observe the boundary behavior $\lim_{k \to 0} G(k; w) = -\infty$ due to (A.3). By continuity, it suffices to find a $k < w$ such that $G(k; w) \geq 0$. Suppose $P \equiv 0$. Then the solution is $\tilde{k} = k_0 := K_0(w)$ defined by (5) and $\tilde{p} = 0$. If $P \neq 0$, consider the following two cases. First, $S(k_0) \geq w$. Then, by (A.4) and monotonicity and continuity of S, there exists a unique value $0 < \hat{k} \leq k_0$ such that $S(\hat{k}) = w$ which implies $\lim_{k \nearrow \hat{k}} G(k; w) = \infty$. Second, suppose $S(k_0) < w$. Then, $\lim_{k \nearrow k_0} G(k; w) = u'(w - S(k_0)) - D(k_0) \geq G_0(k_0; w) = 0$ with G_0 defined by (5). Thus, in either case, there exists a solution $0 < \tilde{k} \leq k_0 < w$. Setting $\tilde{p} = \tilde{P}(\tilde{k})$ completes the proof. ■

A.2 Proof of Lemma 2.2

Let $P \in \mathscr{G}$ be arbitrary. As shown in the previous proof, $TP = \tilde{P} \circ K_P$ where $\tilde{P}$ is defined in (A.1) and, for $w \in \mathbb{W}$, $k = K_P(w)$ is the unique solution to $G(k; w) = 0$ defined in (A.5). Clearly, K_P is continuous. Note from (A.1) that $TP \geq 0$, $P > 0$ implies $TP > 0$ and $P = 0$ implies $TP = 0$. As G in (A.5) is increasing in P and $\tilde{P}$, $K_P \leq K_0$ for all P with strict inequality if $P > 0$. By definition of K_P and (A.5), $w > S(K_P(w)) > \tilde{P}(K_P(w)) = TP(w)$ for $w \in \mathbb{W}$ which proves $TP < \mathrm{id}_{\mathbb{W}}$.

To show that $w \mapsto w - TP(w)$ is (even strictly) increasing, let $w \in \mathbb{W}$ be arbitrary and choose $\Delta > 0$ such that $w + \Delta \in \mathbb{W}$. We show that $TP(w + \Delta) < TP(w) + \Delta$. By contradiction, suppose $TP(w + \Delta) \geq TP(w) + \Delta$. Note that G defined in (A.5) is strictly decreasing in w and strictly increasing in k by strict monotonicity of D and S. These properties imply that K_P is strictly increasing which gives $K_P(w + \Delta) > K_P(w)$. Further, as shown in the previous proof, the function D defined in (A.2) is strictly decreasing which gives $D(K_P(w + \Delta)) < D(K_P(w))$. But by (A.5) and our hypothesis

$$\begin{aligned} D(K_P(w + \Delta)) &= u'(w + \Delta - TP(w + \Delta) - K_P(w + \Delta)) \\ &\geq u'(w - TP(w) - K_P(w + \Delta)) \\ &> u'(w - TP(w) - K_P(w)) \\ &= D(K_P(w)) \end{aligned}$$

which is a contradiction and proves that $w \mapsto w - TP(w)$ is increasing.
Next, we show that TP is increasing. As $TP = \tilde{P} \circ K_P$ and we have already shown that

K_P is strictly increasing, it remains to show that $\tilde{P}$ defined in (A.1) is increasing as well. To avoid trivialities, assume in the remainder that $P > 0$. Adjusting the arguments to the case where $P \geq 0$ is straightforward. Let $k \in \mathbb{K}$ and $\Delta > 0$ be arbitrary such that $k + \Delta \in \mathbb{K}$. We show that $\tilde{P}(k + \Delta) \geq \tilde{P}(k)$. By (U1), the map $a \mapsto av'(a + b)$, $a > 0$ is increasing for any $b \geq 0$. Using this and monotonicity of $P \circ W$ and v' in (A.1) gives

$$\tilde{P}(k + \Delta) \geq V(\Delta) := \frac{\mathbb{E}_\nu\big[P(k,\cdot)v'\big(P(k,\cdot) + (k + \Delta)R(k + \Delta,\cdot)\big)\big]}{\mathbb{E}_\nu\big[R(k + \Delta,\cdot)v'\big(P(k,\cdot) + (k + \Delta)R(k + \Delta,\cdot)\big)\big]} =: \frac{V_1(\Delta)}{V_2(\Delta)}$$

where we abuse our notation by writing just $P(k, \theta)$ rather than $P(W(k, \theta))$. As $V(0) = \tilde{P}(k)$, it suffices to show that V is increasing if either (T2) or (U2) holds. Observe that V is C^1 and the derivative satisfies $V'(\Delta) \geq 0$, if and only if

$$(k + \Delta)V_1'(\Delta)V_2(\Delta) - (k + \Delta)V_2'(\Delta)V_1(\Delta) \geq 0. \tag{A.6}$$

The derivatives in (A.6) compute as

$$V_1'(\Delta) = -(1 - E_{f'}(k + \Delta))\mathbb{E}_\nu\big[B(k,\cdot)R(k + \Delta;\cdot)|v''(-)|\big] \tag{A.7}$$

$$V_2'(\Delta) = -\frac{E_{f'}(k + \Delta)}{k + \Delta}V_2(\Delta) - (1 - E_{f'}(k + \Delta))\mathbb{E}_\nu\big[R(k + \Delta;\cdot)^2|v''(-)|\big]. \tag{A.8}$$

Suppose (T2) holds. Then, (A.7) and (U1) imply $(k+\Delta)V_1'(\Delta) \geq -(1-E_{f'}(k+\Delta))V_1(\Delta)$ while $-(k+\Delta)V_2'(\Delta) \geq E_{f'}(k+\Delta)V_2(\Delta)$ due to (A.8). Using both inequalities together with (T2) shows that (A.6) holds.
Second, suppose (U2) holds and let $\Delta \geq 0$ be fixed. Consider the non-negative random variables $Y := P(k,\cdot)|v''(P(k,\cdot) + (k + \Delta)R(k + \Delta,\cdot))|^{\frac{1}{2}}$ and $X := (k + \Delta)R(k + \Delta,\cdot)|v''(P(k,\cdot)+(k+\Delta)R(k+\Delta,\cdot))|^{\frac{1}{2}}$ both defined on the probability space $(\Theta, \mathscr{B}(\Theta), \nu)$. Then, $V_1(\Delta) = \theta^{-1}\big(\mathbb{E}_\nu[Y^2 + XY]\big)$ and $(k + \Delta)V_2(\Delta) = \theta^{-1}\big(\mathbb{E}_\nu[X^2 + XY]\big)$, $(k + \Delta)V_1'(\Delta) = -(1 - E_{f'}(k + \Delta))\mathbb{E}_\nu[XY]$, and $-(k + \Delta)^2V_1'(\Delta) > (1 - E_{f'}(k + \Delta))\mathbb{E}_\nu[X^2]$. Combining these inequalities shows that (A.6) is satisfied provided that

$$\Big(\mathbb{E}_\nu\big[X^2\big]\Big)^{\frac{1}{2}}\Big(\mathbb{E}_\nu\big[Y^2\big]\Big)^{\frac{1}{2}} \geq \mathbb{E}_\nu\big[|XY|\big]. \tag{A.9}$$

But (A.9) follows from Hölder's inequality (see Aliprantis & Border (2007, p.463 setting $p = q = 2$ which implies $\frac{1}{p} + \frac{1}{q} = 1$)).
Summarizing, we have proved that V is weakly increasing if either (T2) or (U2) hold which implies the desired result

$$\tilde{P}(k + \Delta) \geq V(\Delta) \geq V(0) = \tilde{P}(k).$$

Finally, adopting an argument used and proved in Morand & Reffett (2003, p.1360), monotonicity of TP and $w \mapsto w - TP(w)$, $w \in \mathbb{W}$ imply continuity of TP. ■

A.3 Proof of Lemma 2.3

Let $P \in \mathscr{G}'$ be arbitrary. We need to show that TP is C^1. Since P is C^1, so are the mappings $\tilde{P}$, S, D, and $\tilde{N}$ defined in (A.1) and (A.2) and G defined in (A.5). Recall that for each $w \in \mathbb{W}$, K_P determines the unique zero of $G(\cdot; w)$. Since $\partial_k G(k; w) > 0$, K_P is C^1 by the implicit function theorem. Thus, $TP = \tilde{P} \circ K_P$ is C^1 as well. ■

A.4 Proof of Lemma 2.4

We only prove the strict inequalities, as the proof of the weak inequalities is analogous. Given $P_1, P_0 \in \mathscr{G}'$, suppose $P_1 > P_0$. For $\lambda \in [0, 1]$, define $P_\lambda := \lambda P_1 + (1 - \lambda) P_0$. Since $\mathscr{G}'$ is convex, $P_\lambda \in \mathscr{G}'$ and the derivative satisfies $0 \le P'_\lambda \le 1$ for all λ. Moreover, the map $\lambda \mapsto P_\lambda = P_0 + \lambda\Delta$ where $\Delta := P_1 - P_0 > 0$ is strictly increasing.
Let $w \in \mathbb{W}$ be arbitrary but fixed. By Lemma 2.1 (and a slight abuse of notation), for each $\lambda \in [0, 1]$ there exists a unique pair (k_λ, p_λ) which solves $H_1(k_\lambda, p_\lambda; w, \lambda) = H_2(k_\lambda, p_\lambda; w, \lambda) = 0$. We will now show that $\lambda \mapsto k_\lambda$, $\lambda \in [0, 1]$ is strictly decreasing and $\lambda \mapsto p_\lambda$, $\lambda \in [0, 1]$ is strictly increasing. This implies $p_1 > p_0$ and $k_1 < k_0$ and the claim.

Employing the same definitions and notation as in the proof of Lemma 2.1, write $c_\lambda(k, \theta) := P_\lambda(W(k, \theta)) + kR(k, \theta)$. Then, the pair (k_λ, p_λ) satisfies $p_\lambda = \tilde{P}(k_\lambda, \lambda)$ where

$$\tilde{P}(k, \lambda) := \frac{\mathbb{E}_\nu\big[P_\lambda(W(k, \cdot)) v'\big(c_\lambda(k, \cdot)\big)\big]}{\mathbb{E}_\nu\big[R(k, \cdot) v'\big(c_\lambda(k, \cdot)\big)\big]} =: \frac{N(k, \lambda)}{D(k, \lambda)}, \quad k \in \mathbb{K}, \lambda \in [0, 1]. \tag{A.10}$$

To compute the partial derivatives of D and N, note that $\partial_k W(k, \theta) = E_{f'}(k) R(k, \theta) > 0$ by (1a,b) implies $\partial_k c_\lambda(k, \theta) = R(k, \theta)\big(E_{f'}(k) P'_\lambda(-) + 1 - E_{f'}(k)\big) > 0$. Taking the derivative of (A.10) one obtains, exploiting (U1) and suppressing arguments when convenient

$$\begin{aligned}
\partial_k N(k, \lambda) &= \mathbb{E}_\nu\Big[P'_\lambda(\bullet) E_{f'}(k) R(k, \cdot) v'(\bullet) - P_\lambda(\bullet)|v''(\bullet)|\partial_k c_\lambda(k, \cdot)\Big] && \text{(A.11)}\\
\partial_\lambda N(k, \lambda) &= \mathbb{E}_\nu\big[\Delta(k, \cdot)\big(v'(\bullet) - P_\lambda(W(k, \cdot))|v''(\bullet)|\big)\big] > 0 && \text{(A.12)}\\
\partial_k D(k, \lambda) &= -\frac{1}{k}\mathbb{E}_\nu\Big[E_{f'}(k) R(k, \cdot) v'(\bullet) + kR(k, \cdot)|v''(\bullet)|\partial_k c_\lambda(k, \cdot)\Big] < 0 && \text{(A.13)}\\
\partial_\lambda D(k, \lambda) &= -\mathbb{E}_\nu\big[\Delta(k, \cdot) R(k, \cdot)|v''(-)|\big] < 0 && \text{(A.14)}
\end{aligned}$$

where $\Delta(k, \theta) := P_1(W(k, \theta)) - P_0(W(k, \theta)) > 0$.

We show that $\frac{dk_\lambda}{d\lambda} < 0$. As k_λ is the unique solution to $G(k, \lambda) := u'(w - k - \tilde{P}(k, \lambda)) - D(k, \lambda) = 0$, the implicit function theorem yields the derivative

$$\frac{dk_\lambda}{d\lambda} = -\frac{\partial_\lambda G(k, \lambda)}{\partial_k G(k, \lambda)}\Big|_{k=k_\lambda} = -\frac{|u''(\bullet)|\partial_\lambda \tilde{P}(k_\lambda, \lambda) - \partial_\lambda D(k_\lambda, \lambda)}{|u''(\bullet)|(1 + \partial_k \tilde{P}(k_\lambda, \lambda)) - \partial_k D(k_\lambda, \lambda)}. \tag{A.15}$$

By (U1) and strict monotonicity of c_λ, the map $S(k, \lambda) := k + \tilde{P}(k, \lambda)$ satisfies $\partial_k S(k, \lambda) = 1 + \partial_k \tilde{P}(k, \lambda) > 0$. Further, combining (A.10) with (A.12) and (A.14) shows that $\partial_\lambda \tilde{P}(k, \lambda) > 0$. Using these results with (A.13) and (A.14) in (A.15) gives $\frac{dk_\lambda}{d\lambda} < 0$.

Second, we show that $\frac{dp_\lambda}{d\lambda} > 0$. As $p_\lambda = \tilde{P}(k_\lambda, \lambda)$ one obtains the derivative

$$\frac{dp_\lambda}{d\lambda} = \partial_k \tilde{P}(k_\lambda, \lambda)\frac{dk_\lambda}{d\lambda} + \partial_\lambda \tilde{P}(k_\lambda, \lambda). \tag{A.16}$$

Using (A.15), the derivative (A.16) can equivalently be written as

$$\frac{dp_\lambda}{d\lambda} = \frac{|u''(\bullet)|\partial_\lambda \tilde{P}(k_\lambda, \lambda) + M(k_\lambda, \lambda)}{|u''(\bullet)|(1 + \partial_k \tilde{P}(k_\lambda, \lambda)) - \partial_k D(k_\lambda, \lambda)} \tag{A.17}$$

where $M(k, \lambda) := \partial_\lambda D(k, \lambda)\partial_k \tilde{P}(k, \lambda) - \partial_k D(k, \lambda)\partial_\lambda \tilde{P}(k, \lambda)$. By (A.13) and our previous result, both the denominator and the first term in the numerator in (A.17) are strictly positive. Hence, it suffices to show that $M(k_\lambda, \lambda) \geq 0$. Using the explicit form of the derivatives $\partial_k \tilde{P}$ and $\partial_\lambda \tilde{P}$ computed from (A.10), this last expression can be written as

$$M(k, \lambda) = \frac{\partial_\lambda D(k, \lambda)\partial_k N(k, \lambda) - \partial_k D(k, \lambda)\partial_\lambda N(k, \lambda)}{D(k, \lambda)}.$$

Using (U1), (A.12), and (A.14) gives $\partial_\lambda N(k, \lambda) \geq -k\partial_\lambda D(k, \lambda)$. Thus, it suffices to show $\partial_k N(k, \lambda) + k\partial_k D(k, \lambda) \leq 0$. By (A.11) and (A.13), recalling that $0 \leq P'_\lambda \leq 1$,

$$\partial_k N(k, \lambda) < \mathbb{E}_\nu \Big[P'_\lambda(\bullet)E_{f'}(k)R(k, \cdot)v'(\bullet)\Big] \leq \mathbb{E}_\nu \Big[E_{f'}(k)R(k, \cdot)v'(\bullet)\Big] = -k\partial_k D(k, \lambda).$$

This shows that $M(k_\lambda, \lambda) \geq 0$ and proves the claim. ■

A.5 Proof of Corollary 2.1

(i)$T_d P_1 = T(P_1 + d) \geq T(P_0 + d) = T_d P_0$. (ii)$T_{d_1} P = T(P + d_1) \geq T(P + d_0) = T_{d_0} P$. ■

A.6 Proof of Theorem 1

(i) We show the fixed point property for $d = 0$. The proof for $d > 0$ is analogous. For convenience, we drop the subscript $d = 0$ and denote the sequence $(T^n P_0)_{n \geq 0}$ simply as $(P_n)_{n \geq 0}$ and its pointwise limit by P^*. Also, for the sake of brevity we abuse our notation by writing $P(k, \theta)$ instead of $P(W(k, \theta))$.

Let $w \in \mathbb{W}$ be arbitrary but fixed. As $(P_n)_n$ is a decreasing sequence of functions in $\mathscr{G}'$, monotonicity of $K_\bullet$ due to Lemma 2.4 implies that the sequence $k_n := K_{P_n}(w)$, $n \geq 0$ is strictly increasing and converges to some limit $0 < k^* \leq K_0(w) \leq k_{\max}$. The claim will follow if we show that k^* and $p^* := P^*(w)$ satisfy (12), i.e., $H_1(k^*, p^*; w, P^*, 0) = H_2(k^*, p^*; w, P^*, 0) = 0$. Uniqueness of the solution to (12) then implies $k^* = K_{P^*}(w)$.

Let $\theta \in [\theta_{\min}, \theta_{\max}]$ be arbitrary but fixed. We show that $\lim_{n\to\infty} P_n(k_n, \theta) = P^*(k^*, \theta)$. As $(P_n)_{n \geq 0}$ is a sequence of increasing functions which converges pointwise to the continuous function P^*, convergence is uniform on $\overline{\mathbb{W}} := [W(k_0, \theta_{\min}), w_{\max}] \subset \mathbb{W}$ by Theorem A in Buchanan & Hildebrandt (1908). Note that $W(k_n, \theta) \in \overline{\mathbb{W}}$ for $n \geq 0$.

Thus, for each $\delta > 0$, there is $n_0 \geq 0$ such that $|P_n(k_n, \theta) - P^*(k_n, \theta)| < \delta/2$ for all $n \geq n_0$. Further, by continuity of W and P^* there is $n_0' > 0$ such that $n \geq n_0'$ implies $|P^*(k_n, \theta) - P^*(k^*, \theta)| < \delta/2$. Combining both insights, we have for all $n \geq \max\{n_0, n_0'\}$:

$$|P_n(k_n, \theta) - P^*(k^*, \theta)| \leq |P_n(k_n, \theta) - P^*(k_n, \theta)| + |P^*(k_n, \theta) - P^*(k^*, \theta)| < \delta.$$

For $\theta \in [\theta_{\min}, \theta_{\max}]$, define the functions $\phi_n^1(\theta) := R(k_n, \theta)v'(P_n(k_n, \theta) + k_n R(k_n, \theta))$ and $\phi_n^2(\theta) := P_n(k_n, \theta)v'(P_n(k_n, \theta) + k_n R(k_n, \theta))$. The previous result and continuity of v' and R imply for each $\theta \in [\theta_{\min}, \theta_{\max}]$

$$\begin{aligned}\lim_{n\to\infty} \phi_n^1(\theta) &= \phi_*^1(\theta) := R(k^*, \theta)v'(P^*(k^*, \theta) + k^* R(k^*, \theta)) \\ \lim_{n\to\infty} \phi_n^2(\theta) &= \phi_*^2(\theta) := P^*(k^*, \theta)v'(P^*(k^*, \theta) + k^* R(k^*, \theta)).\end{aligned}$$

As $\phi_n^1(\theta) < R(k_1, \theta_{\max})v'(k_1 R(k_1, \theta_{\min}))$ and $\phi_n^2(\theta) < w_{\max} v'(k_1 R(k_1, \theta_{\min}))$ for all n, the Lebesgue dominated convergence theorem implies $\lim_{n\to\infty} \mathbb{E}_\nu[\phi_n^i(\cdot)] = \mathbb{E}_\nu[\phi_*^i(\cdot)]$, $i = 1, 2$. This, $\lim_{n\to\infty} P_n(w) = p^*$ and $\lim_{n\to\infty} u'(w - P_n(w) - k_n) = u'(w - p^* - k^*)$ imply that (12) is satisfied. Since w was arbitrary, P^* is a fixed point of T.

That $d > 0$ implies $P_d^* > 0$ follows directly from the Euler equations (11a,b) resp. (12).

To prove the stated properties of P_0^*, we show that $P_0^*(w) = 0$ for *some* $w \in \mathbb{W}$ implies $P_0^*(w) = 0$ for *all* $w \in \mathbb{W}$. Let $w_0 \in \mathbb{W}$ be arbitrary and suppose $P_0^*(w_0) = 0$. If $w_0 = w_{\max}$, the claim follows from monotonicity of P_0^*, so suppose $w_0 < w_{\max}$. By (11b) and (12), $P_0^*(w_0) = 0$ implies $P_0^*(W(K_{P_0^*}(w_0), \theta)) = 0$ ν–a.s. As $\theta_{\max}$ is contained in the support of ν, continuity of P_0^* yields $P_0^*(W(K_0^*(w_0), \theta_{\max})) = 0$. Moreover, (11a) and (12) imply $K_0^*(w_0) = K_0(w_0)$, the latter being defined by (5). Thus, under Assumption 3, $w_1 := W(K_0^*(w_0), \theta_{\max})$ satisfies $w_1 = W(K_0(w_0), \theta_{\max}) > w_0$ and $P_0^*(w_1) = 0$.
Let $w_1 \leq w_n < w_{\max}$ be any value for which $P_0^*(w_n) = 0$. Repeating the previous argument shows that $w_{n+1} := W(K_0^*(w_n), \theta_{\max}) = W(K_0(w_n), \theta_{\max}) > w_n$ and $P_0^*(w_{n+1}) = 0$. Due to Assumption 3, the sequence $(w_n)_{n\geq 1}$ converges monotonically to $w_{\max}$ and $P_0^*(w_n) = 0$ for all $n \geq 1$ implies $P_0^*(w_{\max}) = 0$ due to continuity of P_0^*.

The remaining inequalities follow as limits from the monotonicity of $K_\bullet$ and $T_\bullet$ due to Lemma 2.4 and Corollary 2.1 which imply $P_d^m > P_{d'}^m$ and $K_{P_d^m+d} < K_{P_{d'}^m+d'}$ for all m which must (weakly) also hold in the limit. As for each $w \in \mathbb{W}$, $K_d^*(w)$ is the unique zero of $G_d(k; w) = u'(w - k - P_d^*(w)) - \mathbb{E}_\nu[R(k, \cdot)v'(P_d^*(W(k, \cdot)) + d + kR(k, \cdot))]$ which is strictly increasing in d, the second inequality even holds strictly.

(ii) Follows directly from $P_d^* \in \mathscr{G}$ as shown in the main text and Lemma 2.2 (ii).

(iii) Follows directly from the previous results and Definitions 1 and 2. ■

A.7 Proof of Lemma 2.5

Let $(d_n)_{n\geq 0}$ be a sequence converging monotonically to zero. For each $n \geq 1$, define $(P_{d_n}^m)_{m\geq 1}$ as $P_0 = \mathrm{id}_{\mathbb{W}}$ and $P_{d_n}^m = T_{d_n}^m P_0 \in \mathscr{G}'$ for $m \geq 1$. This sequence is strictly monotonic and converges pointwise to $P^*_{d_n} \in \mathscr{G}$ defined in (15). It follows from Theorem 1 (i) that the sequence of limits $(P^*_{d_n})_{n\geq 1}$ is decreasing such that the limiting function

$$P_0^{**}(w) := \lim_{n\to\infty} P^*_{d_n}(w) \tag{A.18}$$

is well-defined for all $w \in \mathbb{W}$. Denote by P_0^* the limit in (15) for $d = 0$, i.e.,

$$P_0^*(w) = \lim_{m\to\infty} T^m P_0(w) \tag{A.19}$$

for $w \in \mathbb{W}$. We would like to show that $P_0^{**} = P_0^*$.

As T_d is increasing in d by Corollary 2.1, $P^m_{d_n} = T^m_{d_n} P_0 \geq T^m P_0 = P^m_0$ for all m which implies $P^*_{d_n} \geq P^*_0$ for all n. Therefore, $P_0^{**} \geq P_0^*$. We therefore need to show $P_0^{**} \leq P_0^*$.

Suppose $d_n = 0$ for all $n \geq n_0$. In this case $n \geq n_0$ implies $P^m_{d_n} = T^m_{d_n} P_0 = T^m P_0 = P^m_0$ for all $m \geq 1$ and, therefore, $P_0^{**} = P_0^*$. The remainder of the proof therefore assumes that the dividend sequence is strictly positive, i.e., $d_n > 0$ for all n and strictly decreasing.

We first show that P_0^{**} in (A.18) is independent of the particular dividend sequence. For $i = 1, 2$, let $(d^i_n)_{n\geq 1}$ be a strictly positive sequence converging monotonically to zero. Denote by $P_0^{**,i}$ the pointwise limit (A.18) induced by $(d^i_n)_{n\geq 1}$. Now, for each $n \geq 1$ there exists $k \geq 0$ such that $d^1_n > d^2_{n+m}$ for all $m \geq k$. By Theorem 1(i), this implies $P^*_{d^1_n} \geq P^*_{d^2_{n+m}}$ and, therefore, $P^*_{d^1_n}(w) \geq \lim_{m\to\infty} P^*_{d^2_{n+m}}(w) = P_0^{**,2}(w)$ for all $w \in \mathbb{W}$. Since n was arbitrary, $P_0^{**,1} \geq P_0^{**,2}$. Reversing the argument gives $P_0^{**,2} \geq P_0^{**,1}$.

We show that $P > P_0^{**}$ implies $TP > P_0^{**}$ for *any* $P \in \mathscr{G}'$. As $P_0 > P_0^{**}$ and $P_0 \in \mathscr{G}'$, we then obtain by simple induction that $T^m P_0 > P_0^{**}$ for all m which proves $P_0^* \geq P_0^{**}$.

Let $P \in \mathscr{G}'$ satisfy $P > P_0^{**}$ and $\hat{w} \in \mathbb{W}$ be arbitrary. We show $TP(\hat{w}) > P_0^{**}(\hat{w})$.[11] Given $\hat{w}$, define the compact set $\overline{\mathbb{W}}_{\hat{w}} := [W(K_P(\hat{w}), \theta_{\min}), w_{\max}] \subset \mathbb{W}$. We will construct a function $\tilde{P} \in \mathscr{G}'$ such that $P > \tilde{P}$ on $\overline{\mathbb{W}}_{\hat{w}}$. Noting that only the behavior of P and $\tilde{P}$ on the interval $\overline{\mathbb{W}}_{\hat{w}}$ is relevant to compute $TP(\hat{w})$ and $T\tilde{P}(\hat{w})$, the same arguments as in the proof of Lemma 2.4 can then be used to show $TP(\hat{w}) > T\tilde{P}(\hat{w})$.[12]

In order to construct such a $\tilde{P}$, set $\delta := \min_{w\in\overline{\mathbb{W}}_{\hat{w}}}\{P(w) - P_0^{**}(w)\} > 0$. By Theorem A in Buchanan & Hildebrandt (1908), there exists a $d > 0$ such that $\|P^*_d(w) - P_0^{**}(w)\|_\infty < \frac{\delta}{3}$ on $\overline{\mathbb{W}}_{\hat{w}}$ as P^*_d converges montonically to P_0^{**} for $d \searrow 0$ (here $\|\cdot\|_\infty$ denotes the supremum norm). By the same argument, there exists $m \in \mathbb{N}$ such that $\|T^m_d P_0(w) - P^*_d(w)\|_\infty < \frac{\delta}{3}$ on $\overline{\mathbb{W}}_{\hat{w}}$ as $(T^m_d P_0)_{m\geq 0}$ converges pointwise to P^*_d. Define $\tilde{P} := T^m_d P_0$ and note that $\|\tilde{P} - P_0^{**}\|_\infty < \frac{2\delta}{3}$ on $\overline{\mathbb{W}}_{\hat{w}}$. Further, $P_0^{**} < T^{m+1}_{\tilde{d}} P_0 < T_{\tilde{d}} \circ T^m_d P_0$ on $\mathbb{W}$ for any $0 < \tilde{d} < d$. Thus, $P_0^{**} < T_{\tilde{d}}\tilde{P}$ for any $\tilde{d} > 0$ which implies $P_0^{**} \leq T\tilde{P}$. This last results uses that

$$\lim_{n\to\infty} T_{d_n} P(w) = TP(w)$$

[11]If $P_0^{**} \in \mathscr{G}'$, this follows trivially by monotonicity of T and the fixed point property $TP_0^{**} = P_0^{**}$ which can be established as in the proof of Theorem 1. Unfortunately, however, we only know $P_0^{**} \in \mathscr{G}$.

[12]Observe that any convex combination $P_\lambda = \lambda P + (1-\lambda)\tilde{P}$ lies between P and $\tilde{P}$. Therefore, by monotonicity of $K_\bullet$, $W(K_{P_\lambda}(\hat{w}), \theta) \in \overline{\mathbb{W}}_{\hat{w}}$ for all $\theta \in [\theta_{\min}, \theta_{\max}]$.

for all $P \in \mathscr{G}'$, $w \in \mathbb{W}$ and any monotonic sequence $(d_n)_n$ converging to zero.[13] Combining these results we get $TP(\hat{w}) > T\tilde{P}(\hat{w}) \geqslant P_0^{**}(\hat{w})$ for any $\hat{w} \in \mathbb{W}$.

To show that $\lim_{n\to\infty} K_{d_n}^*(w) = K_0^*(w)$ for each $w \in \mathbb{W}$, note that $(K_{d_n}^*(w))_n$ is increasing by Theorem 1 (i) and converges to some limit $k^* \le K_0(w)$. By the same arguments used in the proof of Theorem 1 (i), k^* and $p^* := P_0^*(w)$ satisfy the Euler equations at $P = P_0^*$ and $d = 0$ which implies $k^* = K_{P_0^*}(w)$ by uniqueness of the solution to (12). ■

B Efficiency and Inefficiency of MEA

In this appendix, we review the recursive characterization of interim Pareto optimality for stationary exchange economies obtained in Barbie & Kaul (2015) and adapt their results to characterize the optimality of ME in a stochastic production economy. As large parts of the analysis holds almost unchanged and requires mainly notational changes, we refer at many places the reader to Barbie & Kaul (2015) for the details and proofs and just repeat the core facts. To adapt the results, we need the characterization of interim optimality for production OLG models from Barbie, Hagedorn & Kaul (2007) who extended the pure exchange case in Chattopadhyay & Gottardi (1999).

B.1 Notation and definitions

Let $A = (K, C^y, C^o)$ be a continuous, bounded MEA defined as in Section 3.2 and $\overline{\mathbb{W}} = [\underline{w}, w_{\max}]$ be a stable set of A. Fixing the initial shock $\theta_0 \in \Theta$ permits $\overline{\mathbb{W}}$ to be used as the state space which corresponds to the set S in Barbie & Kaul (2015). To adapt our notation to their setup, note that any two successive states w and w' permit to recover the shock in the second period via $\theta' = w'/W(K(w), 1)$. Thus, define the (modified) pricing kernel $m : \overline{\mathbb{W}} \times \overline{\mathbb{W}} \longrightarrow \mathbb{R}_{++}$

$$m(w, w') := \frac{v'\left(C^o(w, w'/W(K(w), 1))\right)}{u'(C^y(w))}. \tag{B.1}$$

Denote by $\mathscr{B}(\overline{\mathbb{W}})$ the Borel-σ algebra on $\overline{\mathbb{W}}$. As shocks are i.i.d., function K defines a transition probability $Q : \overline{\mathbb{W}} \times \mathscr{B}(\overline{\mathbb{W}}) \longrightarrow [0, 1]$,

$$Q(w, G) := \nu(\{\theta \in \Theta \,|\, W(K(w), \theta) \in G.\}). \tag{B.2}$$

Note that Q has the Feller property since the function $W \circ K$ is continuous. By the change-of variable formula, the inequality (18) can be written as

$$\int_{\overline{\mathbb{W}}} \eta(w') m(w, w') Q(w, dw') > \eta(w). \tag{B.3}$$

[13] To see this, fix $w \in \mathbb{W}$ and let $p_n := T_{d_n}P(w)$ and $k_n := K_{P+d_n}(w)$. By Corollary 2.1 and monotonicity of $K_\bullet$, these sequences converge monotonically to values $p^* \ge 0$ and $k^* > 0$, respectively. As $H^i(k_n, p_n, w, P, d_n) = 0$ for all n and $i = 1, 2$, continuity of H^i implies $H^i(k^*, p^*, w, P, 0) = 0$. Uniqueness of the solution to (12) implies $p^* = TP(w)$ and $k^* = K_P(w)$.

To adapt their formal arguments the remainder follows Barbie, Hagedorn & Kaul (2007) by assuming that the shock-process is finite-valued, i.e., $\Theta = \{\theta_1, \ldots, \theta_N\}$. Thus, if $w_t \in \overline{\mathbb{W}}$ is the state in period t, the are n successive states $w_{t+1} = W(K(w_t), \theta_{t+1})$. If $w' \in \overline{\mathbb{W}}$ is a such a successor, we write $w' \succ w_t$. With this notation, an integral of the form (B.3) can be written as $\sum_{w' \succ w} \eta(w') m(w, w') Q(w, w')$.

Given some initial state $w_0 \in \overline{\mathbb{W}}$, denote by $\mathcal{W}^t(w_0)$ the set of histories $w^t = (w_0, \ldots, w_t)$ observed up to time t, i.e., $w_n \succ w_{n-1}$ for all $n = 1, \ldots, t$. Further, let $\mathcal{W}^\infty(w_0)$ denote the set of all infinite histories $w^\infty = (w_t^\infty)_{t \geq 0}$, i.e., $w_t^\infty \succ w_{t-1}^\infty$ for all $t \geq 1$ and $w_0^\infty = w_0$. For any infinite path $w^\infty \in \mathcal{W}^\infty(w_0)$, denote by $(w^\infty)^t$ the induced history up to time $t \geq 0$ along this path, i.e. $(w^\infty)^t = (w_0^\infty, w_1^\infty, ..., w_t^\infty) \in \mathcal{W}^t(w_0)$.

Similar to Chattopadhyay & Gottardi (1999), define for each $w^t \in \mathcal{W}^t(w_0)$ the set *weights*[14]

$$\mathcal{U}(w^t) = \left\{ \lambda(w^t, w') \in \mathbb{R}_+ \mid w' \succ w_t, \sum_{w' \succ w_t} \lambda(w^t, w') Q(w_t, w') = 1 \right\}.$$

Given some $w_0 \in \overline{\mathbb{W}}$, define $\mathcal{U}^\infty(w_0)$ to be the family of weights $\lambda^\infty = (\lambda(w^t, \cdot))_{t \geq 1}$ where $w^t \in \mathcal{W}^t(w_0)$ and $\lambda(w^t, \cdot) \in \mathcal{U}(w^t)$ for all t.

B.2 Recursive characterization of inefficiency

Barbie, Hagedorn & Kaul (2007) derive a condition for interim Pareto inefficiency in a stochastic Diamond model. For a MEA A which satisfies the restrictions from Lemma 3.1, the necessary part of this result can be stated as follows.

Lemma B.1
If $A = (K, C^y, C^o)$ is inefficient at $w_0 \in \overline{\mathbb{W}}$, there exists a family of weights $\lambda^\infty \in \mathcal{U}^\infty(w_0)$ and a constant $C \geq 0$ such that for each path $w^\infty \in \mathcal{W}^\infty(w_0)$

$$\sum_{i=0}^{\infty} \prod_{j=0}^{i} \frac{\lambda\left((w^\infty)^j, w_{j+1}^\infty\right)}{m\left(w_j^\infty, w_{j+1}^\infty\right)} \leq C. \tag{B.4}$$

As noted in Barbie & Kaul (2015), the condition (B.4) can be restated as a minimax problem. The max-part is taking the supremum over all possible paths, the min-part is taking the infimum over all possible weights. For any $w_0 \in \overline{\mathbb{W}}$, define the value function

$$J^*(w_0) := \inf_{\lambda^\infty \in \mathcal{U}^\infty(w_0)} \sup_{w^\infty \in \mathcal{W}^\infty(w_0)} \left\{ 1 + \sum_{i=0}^{\infty} \prod_{j=0}^{i} \frac{\lambda\left((w^\infty)^j, w_{j+1}^\infty\right)}{m\left(w_j^\infty, w_{j+1}^\infty\right)} \right\}. \tag{B.5}$$

[14] As explained in detail in Barbie & Kaul (2015), the definition of a weight given in Chattopadhyay & Gottardi (1999) (and also in Barbie, Hagedorn & Kaul (2007)) is slightly different from here (and in Barbie & Kaul (2015)). Because Chattopadhyay & Gottardi (1999) use an abstract date-event tree setting without objective probabilities, their definition is without probablities, but equivalent to the one given here.

The next result follows immediately from Lemma B.1 and (B.5).

Corollary B.1
If A is inefficient at $w_0 \in \overline{\mathbb{W}}$, then $J^*(w_0) < \infty$.

Following Barbie & Kaul (2015) we show that (B.5) defines a recursive structure permitting J^* to be computed as a fixed point of some operator Z. For each $w \in \overline{\mathbb{W}}$, denote the set of all *stationary weights*

$$\mathcal{U}(w) = \left\{ \lambda(w, w') \in \mathbb{R}_+ \mid w' \succ w, \sum_{w' \succ w} \lambda(w, w') Q(w, w') = 1 \right\}.$$

Define the operator Z which associates with any nonnegative extended real-valued function $J : \overline{\mathbb{W}} \longrightarrow \mathbb{R}_+ \cup \{+\infty\}$ the new function ZJ defined for all $w \in \overline{\mathbb{W}}$ as

$$ZJ(w) := 1 + \inf_{\lambda(w,\cdot) \in \mathcal{U}(w)} \sup_{w' \succ w} \left\{ \frac{\lambda(w, w')}{m(w, w')} \cdot J(w') \right\}. \tag{B.6}$$

Note that Z is monotonic, i.e., $J_1 \geq J_2$ implies $ZJ_1 \geq ZJ_2$. The operator Z can now be used to compute a value function that solves the functional equation (B.6). Construct the sequence $(J_n)_{n \geq 0}$ of functions J_n defined on $\overline{\mathbb{W}}$ recursively by setting $J_0 \equiv 1$ and $J_n = ZJ_{n-1}$ for $n \geq 1$. For each $w \in \overline{\mathbb{W}}$, define the function

$$J_\infty(w) := \lim_{n \to \infty} J_n(w). \tag{B.7}$$

Note that the pointwise limit in (B.7) exists since the sequence $(J_n)_{n \geq 0}$ is increasing. We now have the following result. The proof is the same as in Barbie & Kaul (2015) for Theorem 1 and Proposition 2 (with the appropriate notational changes).

Lemma B.2
The function J_∞ defined in (B.7) is a fixed point of Z that coincides with the value function J^* defined in (B.5), i.e., $J_\infty = ZJ_\infty = J^*$.

B.3 Proof of Lemma 3.1 (i)

By Corollary B.1, if A is inefficient then $J^*(w_0) < \infty$ for all $w_0 \in \overline{\mathbb{W}}$. Set $\eta(w) := 1/J^*(w)$ for $w \in \overline{\mathbb{W}}$. It follows from the same arguments as in the proofs of Proposition 4 and Theorem 2(a) in Barbie & Kaul (2015) that η is a strictly positive, upper-semicontinuous function which takes values in the unit interval (since $J^* > 1$) and satisfies (B.3) for all $w \in \overline{\mathbb{W}}$. As boundedness of A permits to choose the lower bound $\underline{w}$ arbitrarily small, the previous construction of η can be extended to the entire interval $\mathbb{W} =]0, w_{\max}]$. ■

B.4 Proof of Lemma 3.1 (ii)

In this section we present a new additional sufficient condition under which the function η constructed as in the previous subsection is continuous, not just upper-semicontinuous. We will then argue that this condition is satisfied if the kernel m_A exhibits the monotonicity property required in Lemma 3.1 (ii). We have the following result:

Lemma B.3
Suppose $J^* = J_\infty$ defined in (B.7) is uniformly bounded on $\overline{\mathbb{W}}$, i.e., there exists a constant $M \geq 0$ such $J^*(w) \leq M$ for all $w \in \overline{\mathbb{W}}$. Then $\eta = 1/J^*$ is continuous.

Proof: Construct the sequence $(J_n)_{n\geq 0}$ as above by setting $J_0 \equiv 1$ and $J_n = ZJ_{n-1}$ for $n \geq 1$. Recall that $J_1 > 1 = J_0$ and monotonicity of Z imply that $(J_n)_{n\geq 0}$ is strictly increasing, i.e., $J_n > J_{n-1}$ for all $n \geq 0$. By Lemma B.2, we know that the pointwise limit J^* defined in (B.7) is a fixed point of Z. We will show that under the hypotheses of Lemma B.3, $(J_n)_{n\geq 1}$ is a Cauchy sequence in the space of bounded continuous functions on $\overline{\mathbb{W}}$. As this space is complete, the sequence must converge to some bounded continuous function, which coincides with the pointwise limit J^*.

First, we show that each J_n is of the form $J_n(w) = 1+c_n^*(w)$ for some continuous function $c_n^* : \overline{\mathbb{W}} \longrightarrow \mathbb{R}_+$. Clearly, this holds trivially for $n = 0$ and $c_0^* \equiv 0$. By induction, suppose $J_{n-1}(w) = 1+c_{n-1}^*(w)$ for some $n \geq 1$. For each $w \in \overline{\mathbb{W}}$ and $w' \succ w$, define the function

$$\lambda_n^*(w, w') := \frac{m(w, w')}{J_{n-1}(w')} c_n^*(w) \tag{B.8}$$

where c_n^* is chosen such that $\sum_{w'\succ w} \lambda_n^*(w, w')Q(w, w') = 1$ for all $w \in \overline{\mathbb{W}}$, i.e.,

$$c_n^*(w) := \left[\sum_{w'\succ w} \frac{m(w, w')}{J_{n-1}(w')} Q(w, w') \right]^{-1}. \tag{B.9}$$

Note that λ_n^* is continuous and attains the infimum in (B.6). Hence,

$$J_n(w) = 1 + \max_{w'\succ w} \frac{\lambda_n^*(w, w')}{m(w, w')} J_{n-1}(w') = 1 + c_n^*(w). \tag{B.10}$$

As continuity of c_{n-1}^* implies continuity of c_n^*, this proves that each J_n is continuous and, therefore, bounded on the compact set $\overline{\mathbb{W}}$.

Defining λ_n^* by (B.8) for each $n \geq 1$ we can now use the first equality in (B.10) to expand J_n for all $w_0 \in \overline{\mathbb{W}}$ as

$$\begin{aligned} J_n(w_0) &= 1 + \max_{w_1\succ w_0} \frac{\lambda_n^*(w_0, w_1)}{m(w_0, w_1)} \left[1 + \max_{w_2\succ w_1} \frac{\lambda_{n-1}^*(w_1, w_2)}{m(w_1, w_2)} J_{n-2}(w_2) \right] \\ &= 1 + \max_{w_1\succ w_0} \frac{\lambda_n^*(w_0, w_1)}{m(w_0, w_1)} \left[1 + \max_{w_2\succ w_1} \frac{\lambda_{n-1}^*(w_1, w_2)}{m(w_1, w_2)} \left[\cdots \right.\right. \\ &\qquad \left.\left. \left[1 + \max_{w_n\succ w_{n-1}} \frac{\lambda_1^*(w_{n-1}, w_n)}{m(w_{n-1}, w_n)} \right] \cdots \right] \right]. \end{aligned} \tag{B.11}$$

The final term in (B.11) satisfies $1+\max_{w_n \succ w_{n-1}} \frac{\lambda_1^*(w_{n-1},w_n)}{m(w_{n-1},w_n)} = J_1(w_{n-1}) = 1 + c_1^*(w_{n-1})$. Clearly, λ_n^* does not necessarily attain the infimum when defining J_{n+1} by (B.6). Therefore, for all $w_0 \in \overline{\mathbb{W}}$, recalling that $J_1(w) = 1 + c_1^*(w)$

$$\begin{aligned} J_{n+1}(w_0) &= 1 + \max_{w_1 \succ w_0} \frac{\lambda_{n+1}^*(w_0,w_1)}{m(w_0,w_1)} J_n(w_1) \\ &\leq 1 + \max_{w_1 \succ w_0} \frac{\lambda_n^*(w_0,w_1)}{m(w_0,w_1)} J_n(w_1) \\ &\leq 1 + \max_{w_1 \succ w_0} \frac{\lambda_n^*(w_0,w_1)}{m(w_0,w_1)} \left[1 + \max_{w_2 \succ w_1} \frac{\lambda_{n-1}^*(w_1,w_2)}{m(w_1,w_2)} \left[\dots \right.\right. \\ & \quad 1 + \max_{w_{n-1} \succ w_{n-2}} \frac{\lambda_2^*(w_{n-2},w_{n-1})}{m(w_{n-2},w_{n-1})} \left[1 + \max_{w_n \succ w_{n-1}} \frac{\lambda_1^*(w_{n-1},w_n)}{m(w_{n-1},w_n)} (1 + c_1^*(w_n)) \right] \dots \left.\left. \right] \right]. \end{aligned} \tag{B.12}$$

By elementary observations[15], the final term in (B.12) satisfies for any $w_{n-2} \in \overline{\mathbb{W}}$

$$\begin{aligned} & \max_{w_{n-1} \succ w_{n-2}} \frac{\lambda_2^*(w_{n-2},w_{n-1})}{m(w_{n-2},w_{n-1})} \left[1 + \max_{w_n \succ w_{n-1}} \frac{\lambda_1^*(w_{n-1},w_n)}{m(w_{n-1},w_n)} (1 + c_1^*(w_n)) \right] \\ \leq & \max_{w_{n-1} \succ w_{n-2}} \frac{\lambda_2^*(w_{n-2},w_{n-1})}{m(w_{n-2},w_{n-1})} \left[1 + \max_{w_n \succ w_{n-1}} \frac{\lambda_1^*(w_{n-1},w_n)}{m(w_{n-1},w_n)} + \max_{w_n \succ w_{n-1}} \frac{\lambda_1^*(w_{n-1},w_n)}{m(w_{n-1},w_n)} c_1^*(w_n) \right] \\ \leq & \max_{w_{n-1} \succ w_{n-2}} \frac{\lambda_2^*(w_{n-2},w_{n-1})}{m(w_{n-2},w_{n-1})} \left[1 + \max_{w_n \succ w_{n-1}} \frac{\lambda_1^*(w_{n-1},w_n)}{m(w_{n-1},w_n)} \right] \\ + & \max_{w_{n-1} \succ w_{n-2}} \frac{\lambda_2^*(w_{n-2},w_{n-1})}{m(w_{n-2},w_{n-1})} \max_{w_n \succ w_{n-1}} \frac{\lambda_1^*(w_{n-1},w_n)}{m(w_{n-1},w_n)} c_1^*(w_n). \end{aligned}$$

Solving (B.12) in this recursive fashion and using (B.11) we obtain for all n and $w_0 \in \overline{\mathbb{W}}$

$$J_{n+1}(w_0) \leq J_n(w_0) + \max_{w_1 \succ w_0} \frac{\lambda_n^*(w_0,w_1)}{m(w_0,w_1)} \cdots \max_{w_n \succ w_{n-1}} \frac{\lambda_1^*(w_{n-1},w_n)}{m(w_{n-1},w_n)} c_1^*(w_n). \tag{B.13}$$

Using (B.8) and (B.10) in (B.13) we obtain for all $n \in \mathbb{N}$ and $w_0 \in \overline{\mathbb{W}}$

$$\begin{aligned} J_{n+1}(w_0) - J_n(w_0) &\leqslant \max_{w_1 \succ w_0} \frac{c_n^*(w_0)}{1 + c_{n-1}^*(w_1)} \cdot \max_{w_2 \succ w_1} \frac{c_{n-1}^*(w_1)}{1 + c_{n-2}^*(w_2)} \dots \max_{w_n \succ w_{n-1}} c_1^*(w_{n-1}) \cdot c_1^*(w_n) \\ &= c_n^*(w_0) \cdot \max_{w_1 \succ w_0} \frac{c_{n-1}^*(w_1)}{1 + c_{n-1}^*(w_1)} \cdots \max_{w_{n-1} \succ w_{n-2}} \frac{c_1^*(w_{n-1})}{1 + c_1^*(w_{n-1})} \cdot \max_{w_n \succ w_{n-1}} c_1^*(w_n). \end{aligned}$$

Since $M \geqslant J^*(w) \geqslant J_n(w) = 1 + c_n^*(w) > c_n^*(w)$ for any $w \in \overline{\mathbb{W}}$ and $n \in \mathbb{N}$, we get

$$0 < J_{n+1}(w) - J_n(w) \leqslant M^2 \cdot \left(\frac{M}{1+M} \right)^{n-1}$$

[15] These are $\max_x\{A(x) + B(x)\} \leq \max_x\{A(x)\} + \max_x\{B(x)\}$ and $\max_x\{A(x)\} \max_{y \in G(x)}\{B(y) + C(y)\} \leq \max_x\{A(x)\} \max_{y \in G(x)}\{B(y)\} + \max_x\{A(x)\} \max_{y \in G(x)}\{C(y)\}$ for real-valued functions A, B, C and some correspondence G.

for all $w \in \overline{\mathbb{W}}$. But this means that

$$\|J_{n+1} - J_n\|_\infty \leqslant B\,(\beta)^{n-1}$$

where $\|\cdot\|_\infty$ is the supremum norm on the space of bounded continuous functions on $\overline{\mathbb{W}}$ and $B > 0$ and $0 < \beta < 1$. By standard arguments, this implies

$$\|J_{n+m} - J_n\|_\infty \leqslant B\beta^{n-1}\frac{1}{1-\beta}$$

for all $n, m > 0$ and so $(J_n)_{n\geq 0}$ is a Cauchy sequence, as was to be shown. □

Now suppose m_A defined in (17) is monotonically increasing. We show that this implies the hypothesis of Lemma B.3. Using the change of variable formula in (B.9) yields

$$\frac{1}{c_n^*(w)} = \sum_{w' \succ w} \frac{m(w,w')}{1 + c_{n-1}^*(w')} Q(w,w') = \mathbb{E}_\nu\left[\frac{m_A(w,\cdot)}{1 + c_{n-1}^*(W(K(w),\cdot))}\right].$$

As the term to the far right is a strictly increasing function whenever c_{n-1}^* is decreasing, it follows by induction that each $J_n(w) = 1 + c_n^*(w)$, $w \in \overline{\mathbb{W}}$ is strictly decreasing which implies $J_n(w) \leq J_n(\underline{w})$ for all n. Taking the limit gives $J^*(w) \leq J^*(\underline{w})$ for all $w \in \overline{\mathbb{W}}$. Finally, if A is inefficient at w_0, monotonicity of J^* implies $J^*(w_0') \leq J^*(w_0) < \infty$ also for $w_0 \geq w_0'$, i.e., A is also inefficient for all $w_0 \geq w_0'$. ■

B.5 Proof of Lemma 3.2

For each $w \in \mathbb{W}$ and $\theta \in \Theta$, define $C_0^o(w,\theta) := K_0(w)R(K_0(w),\theta)$ and $\tilde{m}(w) := \mathbb{E}_\nu[R(K_0(w),\cdot)v'(C_0^o(w;\cdot))]$. Using (5), the pricing kernel m_0 can be written as

$$m_0(w,\theta) = v'(C_0^o(w,\theta))/\tilde{m}(w).$$

Let $w \in \mathbb{W}$ and $\theta \in \Theta$ be arbitrary but fixed and set $c_0 := C_0^o(w,\theta)$ and $k_0 := K_0(w)$. Then, by direct computations

$$\begin{aligned}\frac{\partial m_0}{\partial w}(w,\theta) = \; & \tfrac{K_0'(w)}{k_0[\tilde{m}(w)]^2}\Big[v''(c_0)c_0(1 - E_{f'}(k_0))\tilde{m}(w) + v'(c_0)E_{f'}(k_0)\tilde{m}(w) \\ & -v'(c_0)(1 - E_{f'}(k_0))\mathbb{E}_\nu[R(k_0,\cdot)C_0(w,\cdot)v''(C_0(w,\cdot))]\Big].\end{aligned}$$

Only the first term in brackets is negative while the other two are strictly positive. Suppose (T2) holds. Then, $1 - E_{f'}(k_0) \leq E_{f'}(k_0)$ and $v''(c_0)c_0 \geq -v'(c_0)$ due to (U1) imply that the first term is dominated by the second one. Suppose (U2) holds. Then, $v''(c_0)c_0 = -\theta v'(c_0)$ and $\mathbb{E}_\nu[R(k_0,\cdot)C_0^o(w,\cdot)v''(C_0^o(w,\cdot))] = -\theta\tilde{m}(w)$ imply that the first term is dominated by the third one. Conclude that $\frac{\partial m_0}{\partial w}(w;\theta) > 0$ in either case. ■

B.6 Proof of Lemma 3.3

As both C^y and C^o are continuous, strictly positive functions on their compact domains $\overline{\mathbb{W}}$ and $\overline{\mathbb{W}} \times \Theta$, respectively, we can choose $\overline{\alpha} > 0$ such that the 'perturbed' allocation $(K, C^y_\alpha, C^o_\alpha)$ defined as $C^y_\alpha(w) := C^y(w) - \alpha\eta(w)$ and $C^o_\alpha(w,\theta) = C^o(w,\theta) + \alpha\eta(W(K(w),\theta))$ is strictly positive and feasible for all $\alpha \in [-\overline{\alpha}, \overline{\alpha}]$ and $w \in \overline{\mathbb{W}}$. Thus, given $w \in \overline{\mathbb{W}}$, the map $h(\alpha; w) := u(C^y_\alpha(w)) + \mathbb{E}_\nu[v(C^o_\alpha(w,\cdot))]$ is well-defined and determines the utility of a generation born in state $w \in \overline{\mathbb{W}}$ under the perturbation $\alpha \in [-\overline{\alpha}, \overline{\alpha}]$. We will determine $\alpha^* > 0$ such that $h(\alpha^*; w) - h(0; w) > 0$ for all $w \in \overline{\mathbb{W}}$, i.e., the perturbed allocation improves the utility of any generation. Let $w \in \overline{\mathbb{W}}$ be fixed. As $h(\cdot; w)$ is twice continuously differentiable on the open interval $]-\overline{\alpha}, \overline{\alpha}[$, we have

$$h(\alpha; w) - h(0; w) = h'(0; w)\alpha + \frac{1}{2}h''(\xi; w)\alpha^2$$

for $0 \le \alpha \le \overline{\alpha}$ and some $0 < \xi < \alpha$ that may depend on both w and α. By hypothesis,

$$h'(0; w) = -u'(C^y(w))\eta(w) + \mathbb{E}_\nu[v'(C^o(w,\cdot))\eta(W(K(w,\cdot))] > 0$$

for all w. Further, using the Lebesgue-dominated convergence theorem

$$h''(\xi; w) = u''(C^y_\xi(w))(\eta(w))^2 + \mathbb{E}_\nu\left[v''\left(C^o_\xi(w,\cdot)\right)(\eta(W(K(w),\cdot)))^2\right] < 0.$$

By the Lebesgue dominated convergence theorem again, both mappings $w \mapsto h'(0; w)$ and $(\xi; w) \mapsto h''(\xi; w)$ are continuous on $\overline{\mathbb{W}}$ and $[0, \overline{\alpha}] \times \overline{\mathbb{W}}$, respectively. Thus, there exist $\Delta_1 > 0$ and $\Delta_2 < 0$ such that $h(\alpha; w) - h(0; w) \geqslant \Delta_1\alpha + \Delta_2\alpha^2$ for all $w \in \overline{\mathbb{W}}$ and $\alpha \in [0, \overline{\alpha}]$. Choosing $\alpha^* > 0$ sufficiently small therefore ensures that $h(\alpha^*; w) > h(0; w)$ for all $w \in \overline{\mathbb{W}}$. ■

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