Supplementary Material for “On the legitimacy of coercion for the financing of public goods”

Felix Bierbrauer*
University of Cologne

September 23, 2011

1 Characterization of Incentive Compatible Mechanisms

Lemma 1 *For all i, the incentive constraints in (2) hold if the following local incentive constraints are satisfied: For any $l < m$,*

$$\theta^l Q_i(\theta^l) - T_i(\theta^l) \geq \theta^l Q_i(\theta^{l+1}) - T_i(\theta^{l+1}) \ , \tag{1}$$

and, for any $l > 0$,

$$\theta^l Q_i(\theta^l) - T_i(\theta^l) \geq \theta^l Q_i(\theta^{l-1}) - T_i(\theta^{l-1}) \ . \tag{2}$$

Moreover, the local incentive constraints (1) and (2) imply that, for all i, $Q_i(\theta^l) \geq Q_i(\theta^{l-1})$, for all $l > 1$.

Proof We first show that for each i and for each l, $Q_i(\theta^l) \geq Q_i(\theta^{l-1})$. This follows from adding (1) for $\theta_i = \theta^l$ (as stated in the Lemma) and (2) for $\theta_i = \theta^{l+1}$,

$$\theta^{l+1} Q_i(\theta^{l+1}) - T_i(\theta^{l+1}) \geq \theta^{l+1} Q_i(\theta^l) - T_i(\theta^l) \ .$$

We now show that (1) implies that

$$\theta^l Q_i(\theta^l) - T_i(\theta^l) \geq \theta^l Q_i(\theta^{l+2}) - T_i(\theta^{l+2}) \ , \tag{3}$$

To see this, rewrite (1) as

$$\theta^l Q_i(\theta^l) - T_i(\theta^l) \geq \theta^{l+1} Q_i(\theta^{l+1}) - T_i(\theta^{l+1}) - (\theta^{l+1} - \theta^l) Q_i(\theta^{l+1}) \ ,$$

Since $Q_i(\theta^{l+2}) \geq Q_i(\theta^{l+1})$ we also have

$$\theta^l Q_i(\theta^l) - T_i(\theta^l) \geq \theta^{l+1} Q_i(\theta^{l+1}) - T_i(\theta^{l+1}) - (\theta^{l+1} - \theta^l) Q_i(\theta^{l+2}) \ ,$$

*University of Cologne, Center for Macroeconomic Research, Albert-Magnus- Platz, 50923 Cologne, Germany. Email: bierbrauer@wiso.uni-koeln.de

Moreover, condition (1) for $\theta_i = \theta^{l+1}$ is

$$\theta^{l+1} Q_i(\theta^{l+1}) - T_i(\theta^{l+1}) \geq \theta^{l+1} Q_i(\theta^{l+2}) - T_i(\theta^{l+2}) \; ,$$

Adding the last two inequalities yields

$$\theta^l Q_i(\theta^l) - T_i(\theta^l) \geq \theta^l Q_i(\theta^{l+2}) - T_i(\theta^{l+2})$$

Hence, an individual with preference parameter θ^l does not benefit from announcing θ^{l+2}. Iterating this argument one more establishes that this individual does neither benefit from announcing θ^{l+3}, etc.

The proof that an individual with preference parameter θ^l does not benefit from announcing θ^{l-j} for any $j \geq 1$ is analogous and left to the reader. ■

Lemma 2 *Suppose that, for some individual i, all local downward incentive compatibility constraints are binding and that $Q_i(\theta^l) \geq Q_i(\theta^{l-1})$, for all $l > 1$. Then all incentive compatibility constraints of individual i are satisfied.*

Proof If all local downward incentive constraints are binding for individual i, this implies that, for all $l \geq 1$,

$$T_i(\theta^l) = \sum_{k=1}^{l} \theta^k (Q_i(\theta^k) - Q_i(\theta^{k-1})) + T_i(\theta^0) \; . \tag{4}$$

Using that $\theta^0 = 0$ and that $\theta^{l+1} - \theta^l = 1$, for all $l > 0$, equation (4) can be equivalently written as

$$T_i(\theta^l) = \theta^l Q_i(\theta^l) - \sum_{k=0}^{l-1} Q_i(\theta^k) + T_i(\theta^0) \; . \tag{5}$$

To establish incentive compatibility, Lemma 1 implies that it suffices to show that all local upward incentive compatibility constraints are satisfied, i.e., for all l,

$$\theta^l Q_i(\theta^l) - T_i(\theta^l) \geq \theta^l Q_i(\theta^{l+1}) - T_i(\theta^{l+1}) \; .$$

or, equivalently,

$$\theta^l Q_i(\theta^l) - T_i(\theta^l) \geq \theta^{l+1} Q_i(\theta^{l+1}) - T_i(\theta^{l+1}) - Q_i(\theta^{l+1}) \; .$$

By equation (5), this inequality can also be written as

$$\sum_{k=0}^{l-1} Q_i(\theta^k) \geq \sum_{k=0}^{l} Q_i(\theta^k) - Q_i(\theta^{l+1}) \; ,$$

or

$$Q_i(\theta^{l+1}) \geq Q_i(\theta^l) \ .$$

These monotonicity constraints are satisfied by assumption. ■

Lemma 3 *If for individual i, all local downward incentive compatibility constraints are binding, then the expected utility of individual i from the ex ante perspective is given by*

$$E[\theta_i q(\theta) - t_i(\theta)] = E\left[\left(\frac{1-P(\theta_i)}{p(\theta_i)}\right) q(\theta)\right] - T_i(\theta^0)$$

Proof Equation (5) in the proof of Lemma 2 and the law of iterated expectations imply that,

$$\begin{aligned}
E[t_i(\theta)] &= \sum_{j=0}^{m} p^j T_i(\theta^j) \\
&= \sum_{j=0}^{m} p^j \theta^j Q_i(\theta^j) - \sum_{j=1}^{m} p^j \sum_{k=0}^{j-1} Q_i(\theta^k) + T_i(\theta^0) \\
&= E[\theta_i q(\theta)] - \sum_{j=1}^{m} \left(1 - \sum_{k=0}^{j} p^j\right) Q_i(\theta^j) + T_i(\theta^0) \\
&= E[\theta_i q(\theta)] - \sum_{j=1}^{m} p^j \, \frac{1 - \sum_{k=0}^{j} p^j}{p^j} \, Q_i(\theta^j) + T_i(\theta^0) \\
&= E[\theta_i q(\theta)] - E\left[\left(\frac{1-P(\theta_i)}{p(\theta_i)}\right) q(\theta)\right] + T_i(\theta^0) \ .
\end{aligned}$$

■

2 Characterization of Revenue Maximizing Mechanisms for a given provision rule

Lemma 4 *For all i, if the participation constraint in (3) is satisfied for $\theta_i = \theta^0$ then it is also satisfied for all $\theta_i \neq \theta^0$.*

Proof Let $\theta_i \neq \theta^0$. Then, by the incentive compatibility constraints in (2),

$$\theta_i Q_i(\theta_i) - T_i(\theta_i) \geq \theta_i Q_i(\theta^0) - T_i(\theta^0) \ .$$

Moreover, $\theta_i > \theta^0$ implies that the right-hand side of this inequality exceeds

$$\theta^0 Q_i(\theta^0) - T_i(\theta^0) \ ,$$

which is nonnegative by the participation constraint for $\theta_i = \theta^0$. This proves that (3) is not binding for $\theta_i \neq \theta^0$. ■

Lemma 5 *Let q be an arbitrary given provision rule. Consider the problem of choosing a mechanism $(t_1, \ldots, t_n)$ in order to maximize total revenue*

$$E\left[\sum_{i=1}^{n} t_i(\theta)\right]$$

subject to subject to the incentive compatibility constraints in (2) and the ex interim participation constraints in (3). At a solution to this problem, the participation constraint in (3) is binding for $\theta_i = \theta^0$ and is slack otherwise.

Proof By Lemma 4 we only need to show that it is binding for $\theta_i = \theta^0$. We show that it is possible to increase the expected payments of individual i in an incentive compatible way if, for some i, the participation constraint for $\theta_i = \theta^0$ does not hold as an equality. It is instructive to rewrite the incentive compatibility constraints in (2) as follows: For each i, for each $\theta_i \in \Theta$, and for each $\hat{\theta}_i \in \Theta$,

$$\theta_i Q_i(\theta_i) - \theta_i Q_i(\hat{\theta}_i) \geq T_i(\theta_i) - T_i(\hat{\theta}_i), \tag{6}$$

Consider a new payment rule for individual i such that for each $\theta_i \in \Theta$, $T_i(\theta_i)$ increases by some $\epsilon > 0$, this implies that the right hand side of the incentive constraints in (6) remains constant, i.e., the increase of i's expected payments does not upset incentive compatibility. Since revenue increase in the expected payments of individual i, the revenue maximizing mechanism must be such that a binding participation constraint for $\theta_i = \theta^0$ prevents a further increase of individual i's payments. ■

Lemma 6 *Let q be an arbitrary given provision rule. Consider the "relaxed problem" of choosing a mechanism $(t_1, \ldots, t_n)$ in order to maximize total revenue*

$$E\left[\sum_{i=1}^{n} t_i(\theta)\right]$$

subject to the downward incentive compatibility constraints in (2) and the ex interim participation constraints in (3). At a solution to this problem, all downward incentive compatibility constraints are binding, and the participation constraint in (3) is binding for $\theta_i = \theta^0$ and is slack otherwise.

Proof It is straightforward to verify that, for all i, all downward incentive compatibility constraints are binding. Otherwise the expected payments of some individual could be increased without violating any one of the constraints of the relaxed problem. It remains to be shown that, for all i, the participation constraint in (3) is binding for $\theta_i = \theta^0$ and is slack otherwise. By Lemma 4 we only need to show that, for all i, the participation constraint in (3) is binding for $\theta_i = \theta^0$. Suppose otherwise. Then it was possible to increase $T_i(\theta^0)$ without violating any constraint. ■

Lemma 7 *Let q be a given provision rule with the property that for all i, and all l, the monotonicity constraints $Q_i(\theta^l) \geq Q_i(\theta^{l-1})$ are satisfied. Consider the problem of choosing $(t_1, \ldots, t_n)$ in order to maximize the total revenue*

$$E\left[\sum_{i=1}^{n} t_i(\theta)\right]$$

subject to the incentive compatibility constraints in (2) and the ex interim participation constraints in (3). The maximal revenue at a solution to this problem is equal to

$$E\left[\sum_{i=1}^{n}\left(\theta_i - \frac{1-P(\theta_i)}{p(\theta_i)}\right) q(\theta)\right].$$

Proof First, consider the "relaxed problem" of maximizing subject to the local downward incentive constraints (2) and the the ex interim participation constraints in (3). The arguments in the proofs of Lemmas 4 - 6 imply that, for all i, all local downward incentive constraints as well as the ex interim participation constraints are binding for $\theta_i = \theta^0$.[1]

Since the given provision rule q satisfies the monotonicity constraints $Q_i(\theta^l) \geq Q_i(\theta^{l-1})$ for all i and l, Lemma 2 implies that all incentive compatibility constraints are satisfied at a solution to the relaxed problem. Hence, the solution to the relaxed problem is the revenue maximizing mechanism.

Given that all local downward incentive compatibility constraints are binding, Lemma 3 implies that, for all i,

$$E[t_i(\theta)] = E\left[\left(\theta_i - \frac{1-P(\theta_i)}{p(\theta_i)}\right) q(\theta)\right] + T_i(\theta^0)$$

Since the participation constraints are binding, for all i, whenever $\theta_i = \theta^0$, we have $T_i(\theta^0) = 0$, for all i, and hence

$$E\left[\sum_{i=1}^{n}\left(\theta_i - \frac{1-P(\theta_i)}{p(\theta_i)}\right) q(\theta)\right].$$

■

3 On the impossibility to reach constrained efficiency subject to interim participation constraints with many individuals

Lemma 8 *Let $(k_n)_{n=1}^{\infty}$ be a sequence of cost functions, with the understanding that k_n is the cost function that applies if the number of individuals is equal to n. Suppose that the sequence $(k_n)_{n=1}^{\infty}$ converges pointwise to a cost function k_∞. Let q_n^* be the surplus maximizing provision rule for an economy with n individuals. Suppose that $(k_n)_{n=1}^{\infty}$ converges pointwise to a cost*

[1] In these Lemmas the provision rule for the public good is not taken as given. However, this does not affect the logic of the argument.

function k_∞ and that

$$\lim_{n\to\infty} \frac{1}{n} E[k_n(q_n^*(\theta))] > 0 \ .$$

Then there exists n', such that for all $n > n'$,

$$\frac{1}{n} E\left[\sum_{i=1}^{n}\left(\theta_i - \frac{1-P(\theta_i)}{p(\theta_i)}\right) q_n^*(\theta)\right] < \frac{1}{n} E[k_n(q_n^*(\theta))] \ .$$

Proof We show in the following that

$$\lim_{n\to\infty} \frac{1}{n} E\left[\sum_{i=1}^{n}\left(\theta_i - \frac{1-P(\theta_i)}{p(\theta_i)}\right) q_n^*(\theta)\right] = 0 \ .$$

By the strong law of large numbers $\frac{1}{n}\sum_{i=1}^{n}\theta_i$ converges in probability to $E[\theta_i]$, and $\frac{1}{n}\sum_{i=1}^{n}\frac{1-P(\theta_i)}{p(\theta_i)}$ converges in probability to $E\left[\frac{1-P(\theta_i)}{p(\theta_i)}\right]$. Moreover, $E\left[\frac{1-P(\theta_i)}{p(\theta_i)}\right] = E[\theta_i]$. To see this, note that

$$E\left[\frac{1-P(\theta_i)}{p(\theta_i)}\right] = \sum_{l=0}^{m} p^l \frac{1-P(\theta^l)}{p^l} = \sum_{l=0}^{m}\sum_{k=l+1}^{m} p_k = \sum_{l=0}^{m} l p^l = \sum_{l=0}^{m} \theta^l p^l \ .$$

Hence,

$$\lim_{n\to\infty} \frac{1}{n} E\left[\sum_{i=1}^{n}\left(\theta_i - \frac{1-P(\theta_i)}{p(\theta_i)}\right)\right] = 0 \ .$$

To complete the proof, note that $q_n^*(\theta)$ converges in probability to a deterministic public goods provision level. This follows since (i) for every given n, $q_n^*(\theta)$ is characterized by the first order condition $\frac{1}{n}\sum_{i=1}^{n}\theta_i = \frac{1}{n}k_n'(q^*(\theta))$, (ii) $\frac{1}{n}\sum_{i=1}^{n}\theta_i$ converges in probability to $E[\theta_i]$, and (iii) $(k_n)_{n=1}^{\infty}$ converges pointwise to k_∞. ∎

By Proposition 2 there is no Pareto-efficient mechanism that satisfies the ex interim participation constraints whenever

$$\frac{1}{n} E\left[\sum_{i=1}^{n}\left(\theta_i - \frac{1-P(\theta_i)}{p(\theta_i)}\right) q_n^*(\theta)\right] < \frac{1}{n} E[k_n(q_n^*(\theta))] \ .$$

In conjunction with Lemma 8 this implies that for sufficiently large n, constrained efficiency is incompatible with the ex interim participation constraints.

4 A model in which tax revenues are distributed ex post

The basic model is such that individuals receive a share $\frac{1-\rho}{n}$ of the expected tax revenue which does not depend on their behavior under the mechanism that the policy-maker proposes. We now investigate an alternative model which is such that individuals receive a share of the profits that the mechanism designer realizes ex post and which therefore depends on the vector of taste parameters θ. This implies that an individual's incentives under the mechanism are complicated by the dependence of profits on the own announcement. The incentive compatibility constraints are now as follows: for each i, for each $\theta_i \in \Theta$, and for each $\hat{\theta}_i \in \Theta$,

$$\theta_i Q_i(\theta_i) - T_i(\theta_i) + \frac{1-\rho}{n} R_i(\theta_i) \geq \theta_i Q_i(\hat{\theta}_i) - T_i(\hat{\theta}_i) + \frac{1-\rho}{n} R_i(\hat{\theta}_i), \tag{7}$$

where

$$R_i(\hat{\theta}_i) := E\left[\sum_{i=1}^{n} t_i(\theta_{-i}, \hat{\theta}_i) - k(q(\theta_{-i}, \hat{\theta}_i)) \mid \hat{\theta}_i\right]$$

are the expected profits from the perspective of individual i.

We want to show in the following that (i) this model is well-defined only if profits are included in the participation constraints i.e., the participation constraints are such that, for all i and all θ_i

$$\theta_i Q_i(\theta_i) - T_i(\theta_i) + \frac{1-\rho}{n} R_i(\theta_i) \geq 0 \tag{8}$$

as opposed to

$$\theta_i Q_i(\theta_i) - T_i(\theta_i) \geq 0 \;, \tag{9}$$

and (ii) that if the model is well defined, then it gives the same results as the basic model under the assumption that $1 - \rho = 0$.

Proposition 7 *The mechanism that maximizes R subject to the incentive constraints in (7), and the interim participation constraints (8) has the following properties.*

i) For all i, the participation constraint for $\theta_i = \theta^0$ is binding and the participation constraints for $\theta_i \neq \theta^0$ are not binding.

ii) For all i, all local downward incentive constraints are binding; i.e., for all $l > 0$,

$$\theta^l Q_i(\theta^l) - T_i(\theta^l) + \frac{1-\rho}{n} R_i(\theta^l) = \theta^l Q_i(\theta^{l-1}) - T_i(\theta^{l-1}) + \frac{1-\rho}{n} R_i(\theta^{l-1}) \;,$$

and all other incentive compatibility constraints are not binding.

iii) The profit maximizing provision rule is given by q_R^{**}.

iv) For all ρ, politician's tax revenue share are given as

$$\rho R^* = E\left[\left(\sum_{i=1}^{n} \theta_i - \frac{1-P(\theta_i)}{p(\theta_i)}\right) q_R^{**}(\theta) - k(q_R^{**}(\theta))\right]$$

v) For all ρ, the expected payoff of individual i from the ex ante perspective, is given by

$$E\left[\left(\frac{1-P(\theta_i)}{p(\theta_i)}\right) q_R^{**}(\theta)\right]$$

Proof of Proposition 7. The proof follows from the series of Lemmas below.

Lemma 9 *For all i, the participation constraint in (8) is binding if $\theta_i = \theta^0$ and is slack otherwise.*

Proof Let $\theta_i \neq \theta^0$. Then, by the incentive compatibility constraints in (7),

$$\theta_i Q_i(\theta_i) - T_i(\theta_i) + \frac{1-\rho}{n} R_i(\theta_i) \geq \theta_i Q_i(\theta^0) - T_i(\theta^0) + \frac{1-\rho}{n} R_i(\theta^0) \, .$$

Moreover, $\theta_i > \theta^0$ implies that the right-hand side of this inequality exceeds

$$\theta^0 Q_i(\theta^0) - T_i(\theta^0) + \frac{1-\rho}{n} R_i(\theta^0) \, ,$$

which is nonnegative by the participation constraint for $\theta_i = \theta^0$, and the fact that the maximal profit-level is non-negative because the policy-maker can always ensure zero profits by choosing $q \equiv 0$. This proves that (8) is not binding for $\theta_i \neq \theta^0$.

Now suppose that $\theta_i = \theta^0$. We show that it is possible to increase the expected payments of individual i in an incentive compatible way if, for some i, the participation constraint for $\theta_i = \theta^0$ does not hold as an equality. It is instructive to rewrite the incentive compatibility constraints in (7) as follows: For each i, for each $\theta_i \in \Theta$, and for each $\hat{\theta}_i \in \Theta$,

$$\theta_i Q_i(\theta_i) - \left(1 - \frac{1-\rho}{n}\right) T_i(\theta_i) + \frac{1-\rho}{n} R_i'(\theta_i) \geq \theta_i Q_i(\hat{\theta}_i) - \left(1 - \frac{1-\rho}{n}\right) T_i(\hat{\theta}_i) + \frac{1-\rho}{n} R_i'(\hat{\theta}_i), \tag{10}$$

where

$$R_i'(\hat{\theta}_i) = E\left[\sum_{j \neq i} t_j(\theta_{-i}, \hat{\theta}_i) - k(q(\theta_{-i}, \hat{\theta}_i)) \mid \hat{\theta}_i\right]$$

does not include the payments of individual i. (10) can equivalently be written as: For each i, for each $\theta_i \in \Theta$, and for each $\hat{\theta}_i \in \Theta$,

$$\theta_i Q_i(\theta_i) + \frac{1-\rho}{n} R_i'(\theta_i) - \theta_i Q_i(\hat{\theta}_i) - \frac{1-\rho}{n} R_i'(\hat{\theta}_i) \geq \left(1 - \frac{1-\rho}{n}\right) (T_i(\theta_i) - T_i(\hat{\theta}_i)), \tag{11}$$

Now fix the provision rule for the public good and the payments of all individuals j, $j \neq i$. This implies that for all of the incentive constraints in (11), the left hand side remains constant. Now, if the policy-maker chooses a new payment rule for individual i such that for each $\theta_i \in \Theta$, $T_i(\theta_i)$ increases by some $\epsilon > 0$, this implies that also the right hand side of the incentive constraints in (11) remains constant, i.e., the increase of i's expected payments does not upset incentive compatibility. Since tax revenues increase in the expected payments of individual i, the tax-revenue-maximizing mechanism must be such that a binding participation constraint for $\theta_i = \theta^0$ prevents a further increase of individual i's payments. ■

Lemma 10 *For all i, the incentive constraints in (7) hold if the following local incentive constraints are satisfied: For any $l < m$,*

$$\theta^l Q_i(\theta^l) - T_i(\theta^l) + \frac{1-\rho}{n} R_i(\theta^l) \geq \theta^l Q_i(\theta^{l+1}) - T_i(\theta^{l+1}) + \frac{1-\rho}{n} R_i(\theta^{l+1}) \, , \tag{12}$$

and, for any $l > 0$,

$$\theta^l Q_i(\theta^l) - T_i(\theta^l) + \frac{1-\rho}{n} R_i(\theta^l) \geq \theta^l Q_i(\theta^{l-1}) - T_i(\theta^{l-1}) + \frac{1-\rho}{n} R_i(\theta^{l-1}) \, . \tag{13}$$

Moreover, the local incentive constraints (12) and (13) imply that, for all i, $Q_i(\theta^l) \geq Q_i(\theta^{l-1})$, for all $l > 1$.

Proof We first show that for each i and for each l, $Q_i(\theta^l) \geq Q_i(\theta^{l-1})$. This follows from adding (12) and the (13) constraint for $\theta_i = \theta^{l+1}$, which is given by

$$\theta^{l+1} Q_i(\theta^{l+1}) - T_i(\theta^{l+1}) + \frac{1-\rho}{n} R_i(\theta^{l+1}) \geq \theta^{l+1} Q_i(\theta^l) - T_i(\theta^l) + \frac{1-\rho}{n} R_i(\theta^l) \, .$$

We now show that (12) implies that

$$\theta^l Q_i(\theta^l) - T_i(\theta^l) + \frac{1-\rho}{n} R_i(\theta^l) \geq \theta^l Q_i(\theta^{l+2}) - T_i(\theta^{l+2}) + \frac{1-\rho}{n} R_i(\theta^{l+2}) \, , \tag{14}$$

To see this, rewrite (12) as

$$\theta^l Q_i(\theta^l) - T_i(\theta^l) + \frac{1-\rho}{n} R_i(\theta^l) \geq \theta^{l+1} Q_i(\theta^{l+1}) - T_i(\theta^{l+1}) + \frac{1-\rho}{n} R_i(\theta^{l+1}) - (\theta^{l+1} - \theta^l) Q_i(\theta^{l+1}) \, .$$

Since $Q_i(\theta^{l+2}) \geq Q_i(\theta^{l+1})$ we also have

$$\theta^l Q_i(\theta^l) - T_i(\theta^l) + \frac{1-\rho}{n} R_i(\theta^l) \geq \theta^{l+1} Q_i(\theta^{l+1}) - T_i(\theta^{l+1}) + \frac{1-\rho}{n} R_i(\theta^{l+1}) - (\theta^{l+1} - \theta^l) Q_i(\theta^{l+2}) \, .$$

Moreover, condition (12) for $\theta_i = \theta^{l+1}$ is

$$\theta^{l+1} Q_i(\theta^{l+1}) - T_i(\theta^{l+1}) + \frac{1-\rho}{n} R_i(\theta^{l+1}) \geq \theta^{l+1} Q_i(\theta^{l+2}) - T_i(\theta^{l+2}) + \frac{1-\rho}{n} R_i(\theta^{l+2}) \, ,$$

Adding the last two inequalities yields

$$\theta^l Q_i(\theta^l) - T_i(\theta^l) + \frac{1-\rho}{n} R_i(\theta^l) \geq \theta^l Q_i(\theta^{l+2}) - T_i(\theta^{l+2}) + \frac{1-\rho}{n} R_i(\theta^{l+2})$$

Hence, an individual with preference parameter θ^l does not benefit from announcing θ^{l+2}. Iterating this argument one more establishes that this individual does neither benefit from announcing θ^{l+3}, etc.

The proof that an individual with preference parameter θ^l does not benefit from announcing θ^{l-j} for any $j \geq 1$ is analogous and left to the reader. ∎

Lemma 11 *At a solution to the policy problem, all local downward incentive compatibility constraints are binding, all other incentive compatibility constraints are not binding, and the expected* *payments of individual i are given by*

$$E[t_i(\theta)] = E\left[\left(\theta_i - \frac{1-P(\theta_i)}{p(\theta_i)}\right) q(\theta)\right] + \frac{1-\rho}{n} R\,. \tag{15}$$

Proof *Step 1.* We first consider the "relaxed problem" of maximizing R subject to the local downward incentive constraints (13), the binding participation constraints for individuals with $\theta_i = \theta^0$, and the following monotonicity constraints: for all i, $Q_i(\theta^l) \geq Q_i(\theta^{l-1})$, for all $l > 1$.

If one of the downward incentive constraints was not binding, then the expected payments of some individual could be increased without violating any one of the constraints of the relaxed problem. Hence, all of these constraints have to be binding. This implies that, for all i and all $l \geq 1$,

$$T_i(\theta^l) = \sum_{k=1}^{l} \theta^k (Q_i(\theta^k) - Q_i(\theta^{k-1})) + \frac{1-\rho}{n}\left(R_i(\theta^l) - R_i(\theta^0)\right) + T_i(\theta^0)\,. \tag{16}$$

From the fact that participation constraints are binding whenever $\theta_i = \theta^0$, it follows that $T_i(\theta^0) = R_i(\theta^0)$. Using that $\theta^0 = 0$ and that $\theta^{l+1} - \theta^l = 1$, for all $l > 0$, equation (16) can be equivalently written as

$$T_i(\theta^l) = \theta^l Q_i(\theta^l) - \sum_{k=0}^{l-1} Q_i(\theta^k) + \frac{1-\rho}{n} R_i(\theta^l)\,. \tag{17}$$

By the law of iterated expectations,

$$\begin{aligned}
E[t_i(\theta)] &= \sum_{j=0}^{m} p^j T_i(\theta^j) \\
&= \sum_{j=0}^{m} p^j \theta^j Q_i(\theta^j) - \sum_{j=1}^{m} p^j \sum_{k=0}^{j-1} Q_i(\theta^k) + \frac{1-\rho}{n} \sum_{j=1}^{m} p^j R_i(\theta^j) \\
&= E[\theta_i q(\theta)] - \sum_{j=1}^{m} \left(1 - \sum_{k=0}^{j} p^j\right) Q_i(\theta^j) + \frac{1-\rho}{n} R \\
&= E[\theta_i q(\theta)] - \sum_{j=1}^{m} p^j \,\frac{1 - \sum_{k=0}^{j} p^j}{p^j}\, Q_i(\theta^j) + \frac{1-\rho}{n} R \\
&= E[\theta_i q(\theta)] - E\left[\left(\frac{1-P(\theta_i)}{p(\theta_i)}\right) q(\theta)\right] + \frac{1-\rho}{n} R\,.
\end{aligned}$$

Step 2. To complete the proof, Lemmas 9-10 imply that it suffices to show that the solution to the relaxed problem satisfies all local upward incentive compatibility constraints, i.e., for all i, and all l,

$$\theta^l Q_i(\theta^l) - T_i(\theta^l) + \frac{1-\rho}{n} R_i(\theta^l) \geq \theta^l Q_i(\theta^{l+1}) - T_i(\theta^{l+1}) + \frac{1-\rho}{n} R_i(\theta^{l+1})\,.$$

or, equivalently,

$$\theta^l Q_i(\theta^l) - T_i(\theta^l) + \frac{1-\rho}{n} R_i(\theta^l) \geq \theta^{l+1} Q_i(\theta^{l+1}) - T_i(\theta^{l+1}) + \frac{1-\rho}{n} R_i(\theta^{l+1}) - Q_i(\theta^{l+1}) \, .$$

By equation (17), this inequality can also be written as

$$\sum_{k=0}^{l-1} Q_i(\theta^k) \geq \sum_{k=0}^{l} Q_i(\theta^k) - Q_i(\theta^{l+1}) \, ,$$

or

$$Q_i(\theta^{l+1}) \geq Q_i(\theta^l) \, .$$

These monotonicity constraints are satisfied at a solution of the relaxed problem. ■

Lemma 12 *At a solution to the policy problem, the politician's tax revenue share are given as*

$$(\rho)R = E\left[\sum_{i=1}^{n} \left(\theta_i - \frac{1-P(\theta_i)}{p(\theta_i)}\right) q(\theta) - k(q(\theta))\right] \, ,$$

and the ex ante expected payoff of individual i equals

$$E\left[\left(\frac{1-P(\theta_i)}{p(\theta_i)}\right) q(\theta)\right]$$

Proof Using Lemma 11 to substitute for $E\left[\sum_{i=1}^{n} t_i(\theta)\right]$ in the definition of R, and collecting terms gives the result for after tax profits. Using Lemma 11 to substitute for $E\left[t_i(\theta)\right]$ in

$$E[\theta^i q(\theta) - t_i(\theta)] + \frac{1-\rho}{n} R$$

gives the result for individual i's expected payoff. ■

Participation Constraints that do not include tax revenues. We now discuss briefly how the analysis changes if instead of the participation constraints in (8) those in (9) that do not include profits, need to be satisfied. The analysis of this problem requires only minor adjustments, relative to the proof of Proposition 7. Adjusting the arguments in the proof of Lemma 9, implies that

$$T_i(\theta^0) = 0. \tag{18}$$

This yields the following modifications in Lemma 11. The expected payments of individual i ex interim, provided that $\theta_i \neq \theta^0$, are now given as

$$T_i(\theta^l) = \theta^l Q_i(\theta^l) - \sum_{k=0}^{l-1} Q_i(\theta^k) + \frac{1-\rho}{n}\left(R_i(\theta^l) - R_i(\theta^0)\right) \tag{19}$$

From the ex ante perspective these expected payments are equal to

$$E[t_i(\theta)] = E\left[\left(\theta_i - \frac{1-P(\theta_i)}{p(\theta_i)}\right) q(\theta)\right] + \frac{1-\rho}{n} R - (1-P(\theta^0))\frac{1-\rho}{n} R_i(\theta^0) \tag{20}$$

Substituting these expected payments into the definition of R implies that

$$(\rho)R = E\left[\sum_{i=1}^{n}\left(\theta_i - \frac{1-P(\theta_i)}{p(\theta_i)}\right)q(\theta) - k(q(\theta))\right] - (1-P(\theta^0))\frac{1-\rho}{n}\sum_{i=1}^{n}R_i(\theta^0)$$

Now it is easy to verify that the mechanism designer may choose the mechanism such that $R(\theta^0)$ is arbitrarily small and hence $E[t_i(\theta)]$ is arbitrarily large even if $T_i(\theta^0) = 0$ and all local downward incentive compatibility constraints are binding.[2] Consequently, after tax profits are unbounded and the profit-maximization problem is no longer well defined.

5 A model in which transfers of tax revenue are included in the participation constraints

The following proposition clarifies how the analysis in Section 4 would be modified if instead of the ex interim participation constraints in (3) that do not include tax revenues the ex interim participation constraints in (7) are imposed.

Proposition 8 *The mechanism that maximizes R subject to the incentive constraints in (2), and the interim participation constraints (7) has the following properties.*

i) The pattern of binding participation and incentive compatibility constraints and the tax-revenue-maximizing provision rule are as in Proposition 7.

ii) The policy-maker's revenue share is, for all ρ, given as

$$\rho R^* = E\left[\left(\sum_{i=1}^{n}\theta_i - \frac{1-P(\theta_i)}{p(\theta_i)}\right)q_R^{**}(\theta) - k(q_R^{**}(\theta))\right] .$$

For all ρ, individual i's ex ante expected payoff is given by,

$$E\left[\frac{1-P(\theta_i)}{p(\theta_i)}q_R^{**}(\theta)\right] .$$

The proposition establishes a neutrality result. The parameter ρ neither has an influence on the optimal mechanism nor on the distribution of payoffs between the policy-maker and the consumers of the public good. The reason is that, whatever ρ, the mechanism designer will ensure that the lowest type's participation constraint is binding, so that

$$T_i(\theta^0) = \frac{1-\rho}{n}R^*$$

for all i. Consequently, a higher $1-\rho$, implies that the mechanism designer can rise the expected payment of individuals with the lowest possible valuation, so that their ex interim expected utility level is equal to 0. Given that the lowest types have an expected utility level of 0, all

[2] Suppose that there are only two individuals, $n = 2$, and only two possible preference parameters, $m = 1$. The policy-maker can choose $q(\theta) = 0$, for all θ and still choose the payments $t_1(\theta)$ and $t_2(\theta)$ such that he makes an unbounded profit, even though incentive and participation constraints have to be satisfied.

higher types can only get the information rent that is due to their private information on their preferences. Hence, whatever $1-\rho$, their expected utility level is equal to $E\left[\frac{1-P(\theta_i)}{p(\theta_i)} q_R^{**}(\theta)\right]$.

An implication of these observations is that the model with tax revenues in the participation constraints yields exactly the same results as the model without tax revenues in the participation constraints under the additional assumption that, in the latter model, $1-\rho = 0$. Hence, without loss of generality, we may focus on the model with participation constraints that do not include tax revenues.[3]

Proof of Proposition 8. The proof of Proposition 8 is very similar to the characterization of the profit maximizing mechanism with ex interim participation constraints in the proof of Proposition 1. In the following, we only sketch the steps where the arguments from the proof of Proposition 1 require some modification. Adjusting the arguments in the proof of Lemma 6, implies that

$$T_i(\theta^0) = \frac{1-\rho}{n} R^*. \tag{21}$$

A straightforward modification of Lemma 7 implies that individual i's expected payments from the ex ante perspective are equal to

$$E[t_i(\theta)] = E\left[\left(\theta_i - \frac{1-P(\theta_i)}{p(\theta_i)}\right) q(\theta)\right] + \frac{1-\rho}{n} R^* \,. \tag{22}$$

Substituting these expected payments into the definition of R yields

$$R = E\left[\left(\sum_{i=1}^{n} \theta_i - \frac{1-P(\theta_i)}{p(\theta_i)}\right) q(\theta) - k(q(\theta))\right] + 1 - \rho R^*$$

After plugging the profit maximizing provision rule q_R^{**} into this formula we obtain, by definition of R^*,

$$R^* = E\left[\left(\sum_{i=1}^{n} \theta_i - \frac{1-P(\theta_i)}{p(\theta_i)}\right) q_R^{**}(\theta) - k(q_R^{**}(\theta))\right] + 1 - \rho R^* \,,$$

or, equivalently,

$$\rho R^* = E\left[\left(\sum_{i=1}^{n} \theta_i - \frac{1-P(\theta_i)}{p(\theta_i)}\right) q_R^{**}(\theta) - k(q^{**}(\theta))\right]$$

Finally, to solve for the the ex ante expected payoff of individual i, rearrange equation (22) to obtain

$$E[\theta_i q(\theta) - t_i(\theta)] + \frac{1-\rho}{n} R^* = E\left[\frac{1-P(\theta_i)}{p(\theta_i)} q(\theta)\right]$$

Upon substituting the tax-revenue-maximizing provision rule q_R^{**} into this formula we obtain the statement in the Proposition. ■

[3]Proposition 8 can also be proven for the model with ex ante instead of ex interim participation constraints. The details are left to the reader.